The Mathematical Association of Victoria

Trial Examination 2024

MATHEMATICAL METHODS

Trial Written Examination 2 - SOLUTIONS

SECTION A: Multiple Choice

Question	Answer	Question	Answer
1	C	11	A
2	C	12	D
3	A	13	C
4	B	14	A
5	B	15	B
6	D	16	D
7	B	17	A
8	D	18	C
9	C	19	B
10	D	20	A

Question 1 **Answer C**

$f(x) = -\frac{3}{2}\sin\left(2x - \pi\right)$

Amplitude: $A = \frac{3}{2}$

Period: $P = \frac{2\pi}{2} = \pi$

Question 2 **Answer C**

$f(x) = \sqrt{x+2}$ and $g(x) = e^{2x}$

Test range of $f \subseteq$ domain of g

$\left[0, \infty\right) \subset R$

domain of $g \circ f =$ domain of $f = \left[-2, \infty\right)$

Question 3 **Answer A**

$0 = ax^2 + 4x + c$

two unique solutions if $\Delta > 0$

$4^2 - 4ac > 0$

$4ac < 16$

$ac < 4$

Question 4 **Answer B**

$x+(m-1)y=2 \Rightarrow y=\left(\dfrac{-1}{m-1}\right)x+\dfrac{2}{m-1}$

$(m+1)x+3y=8-m \Rightarrow y=-\left(\dfrac{m+1}{3}\right)x+\dfrac{8-m}{3}$

Equate gradients

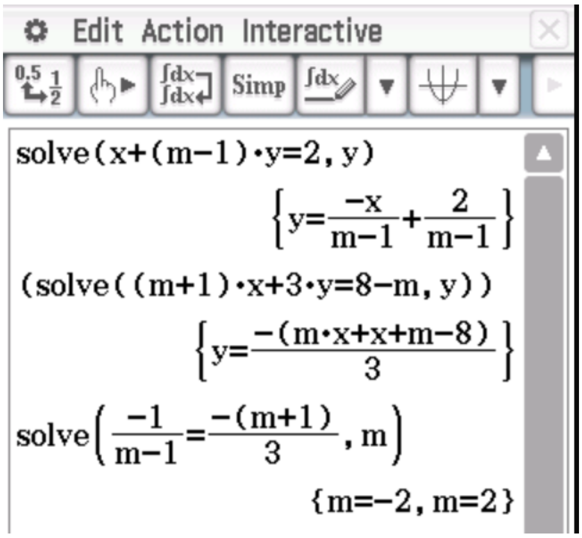
$-\dfrac{1}{m-1}=-\dfrac{m+1}{3}$

Gives $m=\pm 2$

Test for infinite number of solutions

$m=2$	$x+y=2$ $3x+3y=6$
$m=-2$	$x-3y=2$ $-x+3y=10$

Answer: $m=2$



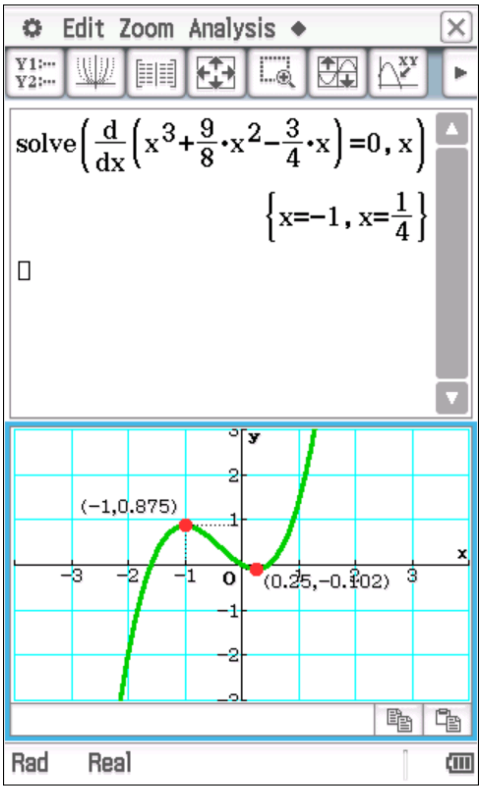
Question 5 **Answer B**

$g(x)=x^3+\dfrac{9}{8}x^2-\dfrac{3}{4}x$

$g'(x)=3x^2+\dfrac{9}{4}x-\dfrac{3}{4}=0$

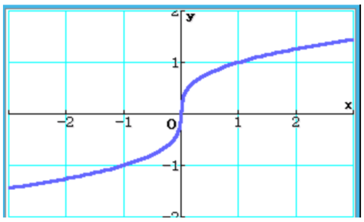
Gives $x=-1,\ x=\dfrac{1}{4}$

strictly decreasing for $\left[-1,\dfrac{1}{4}\right]$



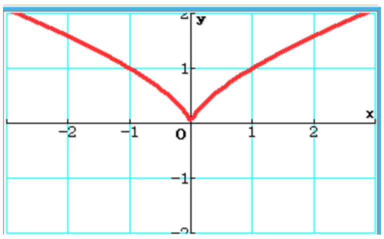
Question 6 **Answer D**

Option A $y=x^{\frac{1}{3}},\ \dfrac{dy}{dx}=\dfrac{1}{3}x^{-\frac{2}{3}}\neq 0$



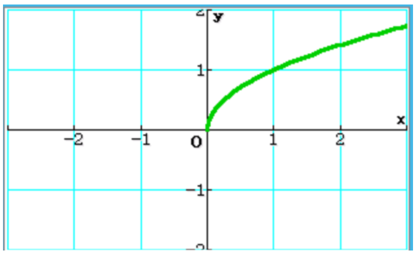
Gradient undefined at $x=0$

Option B $y=x^{\frac{2}{3}},\ \dfrac{dy}{dx}=\dfrac{2}{3}x^{-\frac{1}{3}}\neq 0$



Sharp point at $x=0$

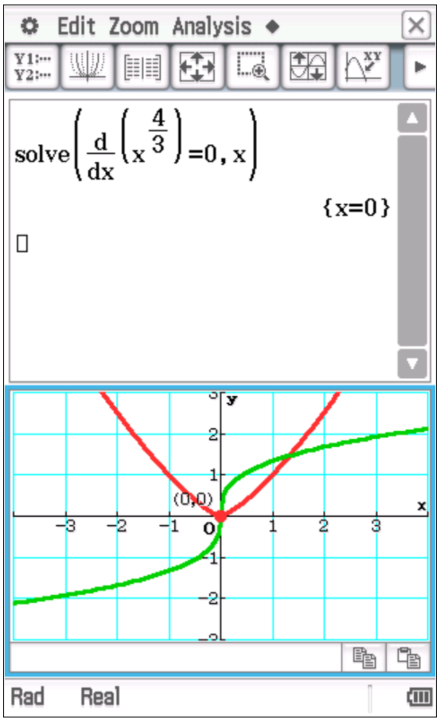
Option C $y=x^{\frac{1}{2}},\ \dfrac{dy}{dx}=\dfrac{1}{2}x^{-\frac{1}{2}}\neq 0$



Endpoint at $(0,0)$

Option D $y = x^{\frac{4}{3}}$, $\frac{dy}{dx} = \frac{4}{3}x^{\frac{1}{3}} = 0$ for $x = 0$

Differentiable for all values over its maximal domain.
Gradient graph exists for all $x \in R$.



Question 7 **Answer B**

Given $\int_1^3 f(x)dx = 4$ and $\int_3^1 g(x)dx = -2$.

Simplify $-\int_1^2 g(x)dx + \int_1^3 (2f(x)+3)dx - \int_2^3 g(x)dx$

$-\left(\int_1^2 g(x)dx + \int_2^3 g(x)dx\right) + 2\int_1^3 f(x)dx + \int_1^3 (3)dx$

$= -\int_1^3 g(x)dx + 2(4) + [3x]_1^3$

$= -2 + 8 + 6$

$= 12$

Question 8 **Answer D**

Two balls of the same colour selected without replacement.

Box A: $\left(\frac{1}{2}\times\frac{4}{7}\times\frac{3}{6}\right)+\left(\frac{1}{2}\times\frac{3}{7}\times\frac{2}{6}\right)=\frac{3}{14}$

Box B: $\left(\frac{1}{2}\times\frac{4}{7}\times\frac{3}{6}\right)+\left(\frac{1}{2}\times\frac{3}{7}\times\frac{2}{6}\right)=\frac{3}{14}$

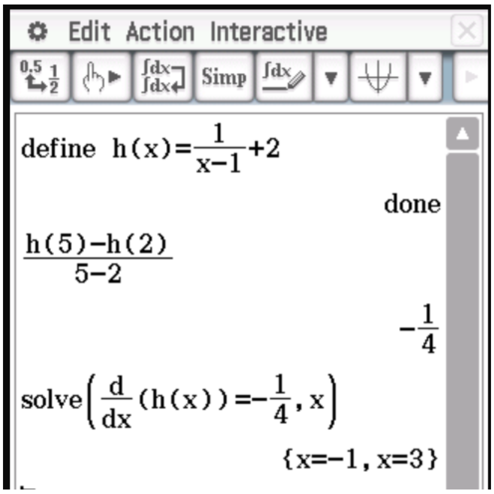
Answer: $\frac{3}{14}+\frac{3}{14}=\frac{3}{7}$

Question 9 **Answer C**

$h: R\setminus\{1\}\to R,\ h(x)=\frac{1}{x-1}+2$.

Average rate of change $=\frac{h(5)-h(2)}{5-2}=-\frac{1}{4}$

$h'(x)=-\frac{1}{4}$ at $x=-1$ or $x=3$

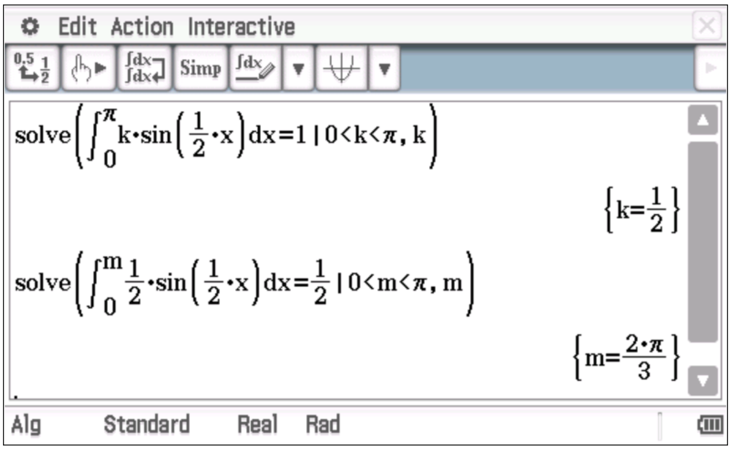


Question 10 **Answer D**

$$f(x)=\begin{cases} k\sin\left(\frac{1}{2}x\right) & 0<x<\pi \\ 0 & \text{otherwise}\end{cases}$$

$\int_0^{\pi} k\sin\left(\frac{1}{2}x\right)=1$ gives $k=\frac{1}{2}$

$\int_0^{m} \frac{1}{2}\sin\left(\frac{1}{2}x\right)=0.5$ gives $m=\frac{2\pi}{3}$



Question 11 Answer A

$f: R\backslash\{1\} \to R,\ f(x) = \dfrac{1}{(x-1)^2} - 2$

The tangent line at $x = 0$ is $y = 2x - 1$.

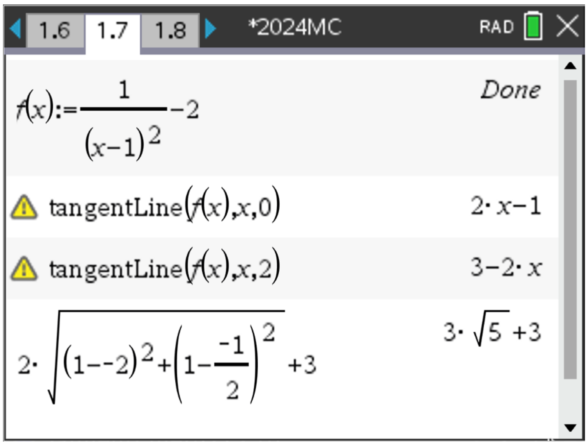
The tangent line at $x = 2$ is $y = 3 - 2x$.

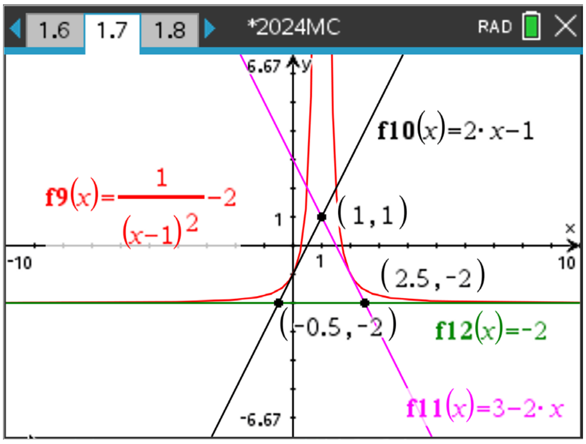
The coordinates of the vertices of the triangle are $(1,1)$, $\left(-\dfrac{1}{2}, -2\right)$ and $\left(\dfrac{5}{2}, -2\right)$.

The length of the base is 3.

The length of each of the other two sides is $\sqrt{\left(1+\dfrac{1}{2}\right)^2 + (1+2)^2} = \dfrac{3\sqrt{5}}{2}$.

Perimeter $= 3\sqrt{5} + 3$





Question 12 Answer D

$y = g(x) = 4\log_2(3x+5)$, $h = 1$

Area of the trapeziums $= \dfrac{1}{2}\big(g(0) + 2g(1) + 2g(2) + g(3)\big)$

$= 2\big(\log_2(5) + \log_2(14) + \log_2(64) + \log_2(121)\big)$

$= 2\big(\log_2(5\times 14\times 121) + \log_2(64)\big)$

$= 2\big(\log_2(5\times 14\times 121) + 6\big)$

$= \log_2(5\times14\times121)^2 + 12$

$\neq \log_2(5\times14\times121)^2 + 6^2$

Question 13 **Answer C**

$f(x) = 3\tan\left(\frac{1}{2}\left(\frac{\pi}{3}x - 1\right)\right) + 5$

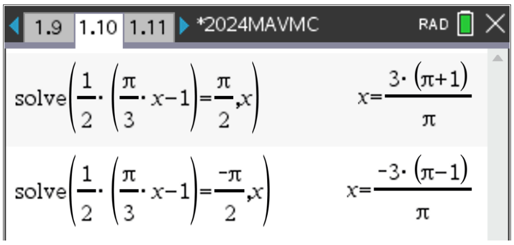
Solve $\frac{1}{2}\left(\frac{\pi}{3}x - 1\right) = \frac{\pi}{2}$, $x = \frac{3(\pi+1)}{\pi}$

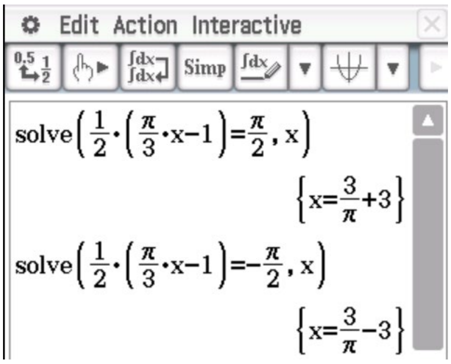
OR

$\frac{1}{2}\left(\frac{\pi}{3}x - 1\right) = -\frac{\pi}{2}$, $x = \frac{-3(\pi-1)}{\pi}$

The period $= \frac{\pi}{\frac{\pi}{6}} = 6$

A general solution is $x = \frac{-3(\pi-1)}{\pi} + 6k,\ k \in Z$





Question 14 **Answer A**

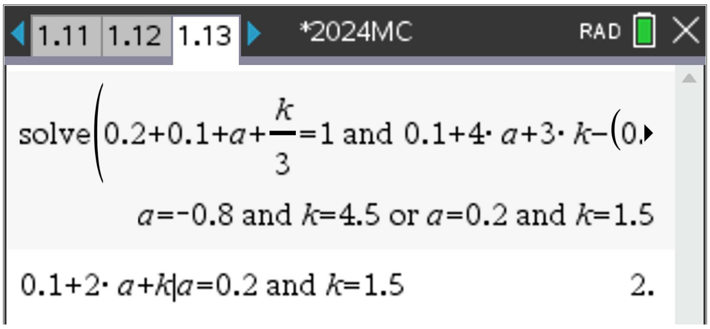
x	0	1	2	3
$\Pr(X = x)$	0.2	0.1	a	$\frac{k}{3}$

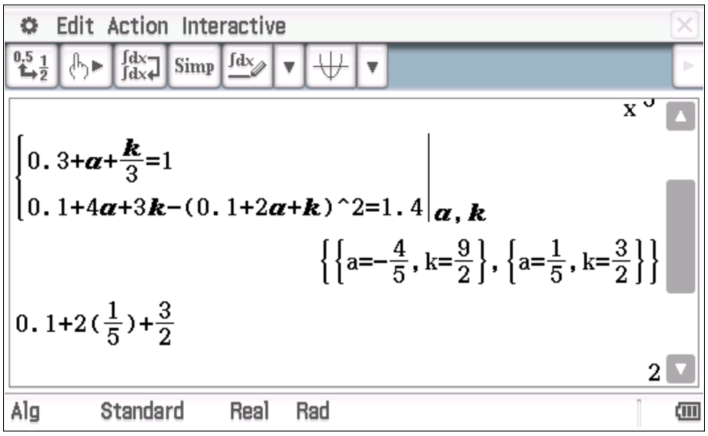
$\text{Var}(X) = 0.1 + 4a + 3k - (0.1 + 2a + k)^2 = 1.4$ …(1)

$0.3 + a + \frac{k}{3} = 1$…(2)

$a = 0.2$, $k = 1.5$

$\text{E}(X) = 0.1 + 2a + k = 2$





Question 15 **Answer B**

The domain of $s(x)=1-\log_e(1-x)$ is $(-\infty,-1)$.

The range $t(x)=3\cos(2x-1)+1$ is $[-2,4]$.

The domain of $s(x)+t^{-1}(x)$ is the intersection of $(-\infty,-1)$ and $[-2,4]$ which is $[-2,1)$.

Question 16 **Answer D**

$f:R\setminus\left\{\frac{a}{4}\right\}\to R, f(x)=\frac{2}{4x-a}+3$

$x_0=\frac{a}{4}$ will fail as $x=\frac{a}{4}$ is an asymptote.

Netwon's method will also fail if the x-intercept of the tangent line at x_n is undefined.

Find the equation of the tangent line at any point on the curve. Let the x-coordinate be *b*.

$$y=\frac{3a^2-2a(12b+1)+16b(3b+1)}{(a-4b)^2}-\frac{8x}{(a-4b)^2}$$

Solve $y=\frac{3a^2-2a(12b+1)+16b(3b+1)}{(a-4b)^2}-\frac{8x}{(a-4b)^2}=0$ when $x=\frac{a}{4}$

$b=\frac{3a-4}{12}$, hence $x_0=\frac{3a-4}{12}$ will fail.

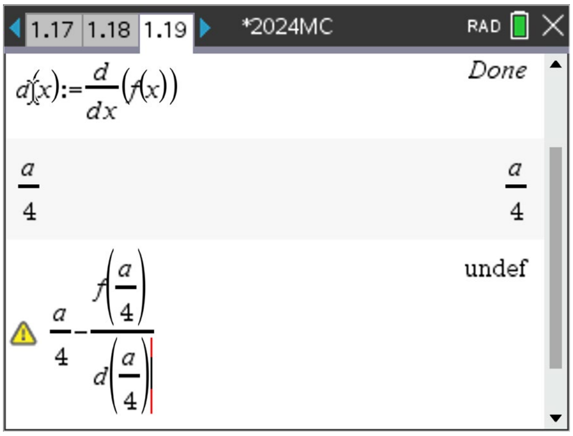
The x_0 values for any point on the RHS branch will fail as none of the x-intercepts of the tangent lines are less than $\frac{a}{4}$.

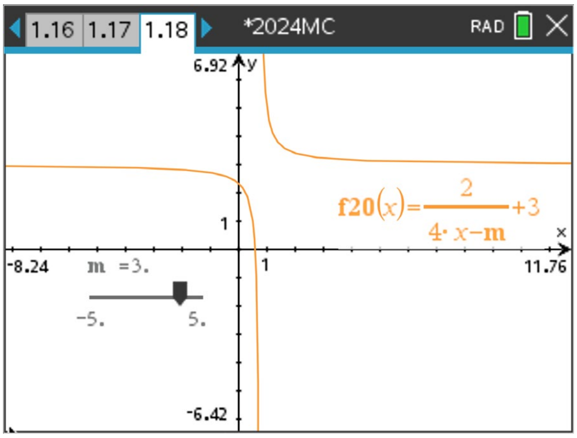
$x_0<\frac{3a-4}{12}$ will also fail as the x-intercept of the tangent lines are all greater than $\frac{a}{4}$.

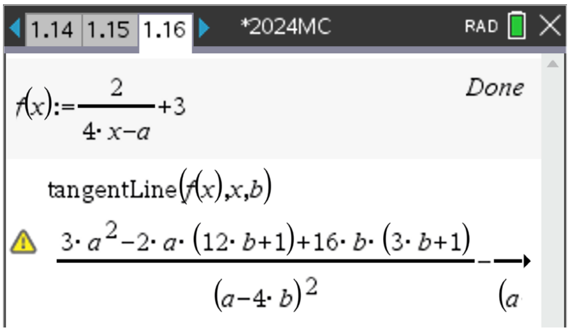
So convergence will only occur if $\frac{3a-4}{12} < x_0 < \frac{a}{4}$.

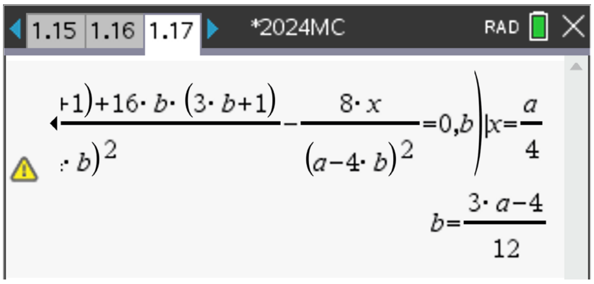
Newton's method fails if $x_0 \in R \setminus \left(\frac{3a-4}{12}, \frac{a}{4}\right)$.

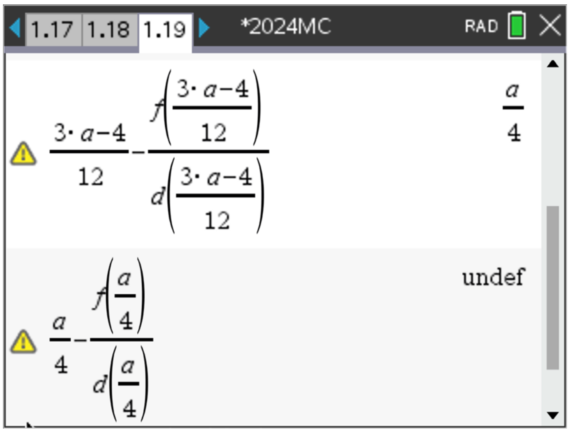
The answer can also be found by checking the values in the options to see if they fail when using Newton's method.











Question 17 **Answer A**

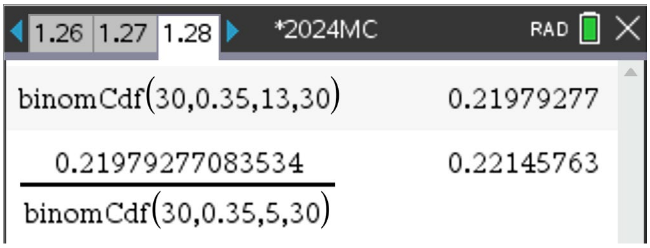
$X \sim \text{Bi}(30, 0.35)$

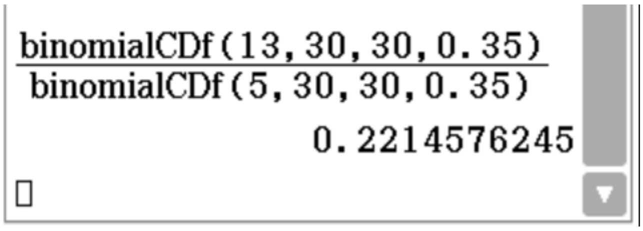
$\Pr(X > 12 \mid X \geq 5)$

$\frac{\Pr(X \geq 13)}{\Pr(X \geq 5)}$

$= \dfrac{0.2197...}{0.9925...}$

$= 0.2215$ correct to four decimal places





Question 18 **Answer C**

Let A_v be the average value of $f(x) = x^3 + x^2 - x + 1$ for the interval $[a,1]$, where $a \in (-\infty,1)$.

$$A_v = \frac{1}{1-a}\int_a^1 f(x)dx = \frac{3a^3 + 7a^2 + a + 13}{12}$$

A_v is a cubic function. $y = A_v$ will have 3 solutions between the two turning points.

Solve $\dfrac{d}{da}\left(\dfrac{3a^3 + 7a^2 + a + 13}{12}\right) = 0$ for *a*.

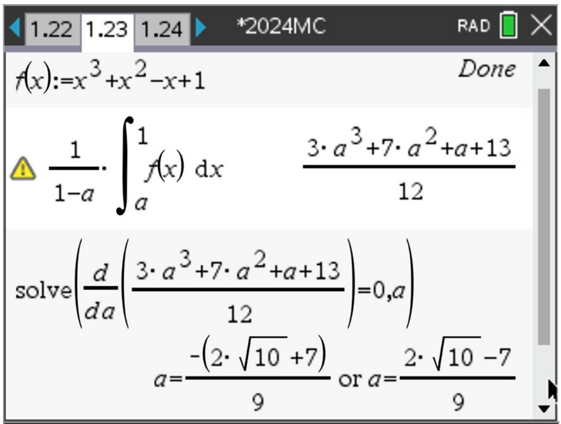
$$a = \frac{2\sqrt{10} - 7}{9}, \quad a = \frac{-2\sqrt{10} - 7}{9}$$

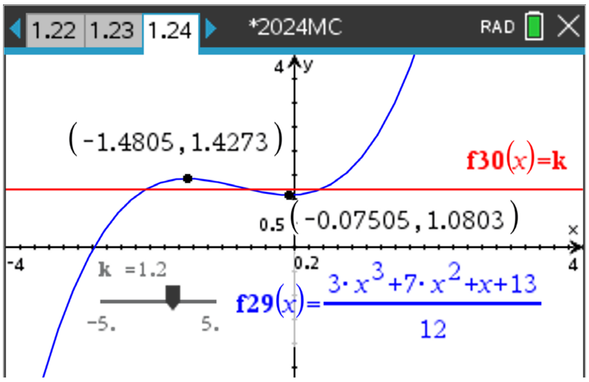
$$A_v\left(\frac{2\sqrt{10} - 7}{9}\right) = \frac{-2\left(20\sqrt{10} - 457\right)}{729}$$

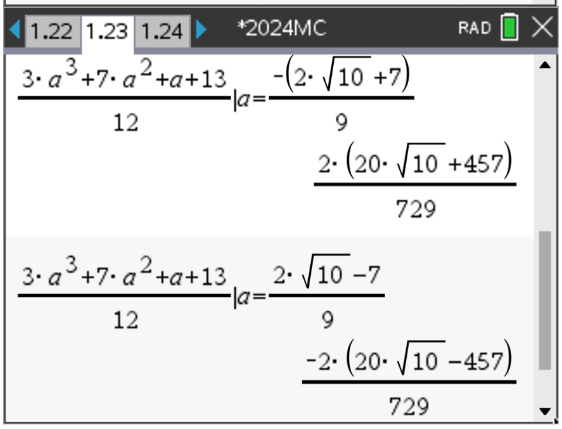
$$A_v\left(\frac{-2\sqrt{10} - 7}{9}\right) = \frac{2\left(20\sqrt{10} + 457\right)}{729}$$

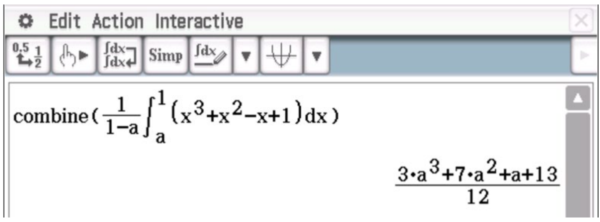
$$A_v \in \left(\frac{-2\left(20\sqrt{10} - 457\right)}{729}, \frac{2\left(20\sqrt{10} + 457\right)}{729}\right)$$

The answer can also be found by checking the values in the options to see if they give three a values.









Question 19 **Answer B**

$$f(x)=\frac{1}{\sqrt{18\pi}}e^{-\frac{1}{2}\left(\frac{2x-3}{6}\right)^2}=\frac{1}{3\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\frac{3}{2}}{3}\right)^2}$$

$$X\sim \mathrm{N}\left(\frac{3}{2},3^2\right)$$

- a dilation by a factor of 3 from the x-axis

$$f_1(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{2x-3}{6}\right)^2}$$

- a dilation by a factor of $\frac{1}{3}$ from the y-axis

$$f_2(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{6x-3}{6}\right)^2}=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(x-\frac{1}{2}\right)^2}$$

- a translation of $\frac{1}{2}$ a unit left.

$$f_3(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$

Check: $(x,y)\to(x,3y)\to\left(\frac{x}{3},3y\right)\to\left(\frac{x}{3}-\frac{1}{2},3y\right)$

$$x'=\frac{x}{3}-\frac{1}{2},\; x=3x'+\frac{3}{2}$$

$y' = 3y,\ y = \dfrac{y'}{3}$

$$\frac{y'}{3} = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{3x'+\frac{3}{2}-\frac{3}{2}}{3}\right)^2}$$

$$y' = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x')^2}$$

Question 20 **Answer A**

$f : R \to R, f(x) = e^{x^3+bx}$

Solve $f''(x) = 0$ but does not work directly on the TI or the CASIO.

So find $f''(x) = \left(9b^2x^4 + 6bx^2 + 6bx + 1\right)e^{x^3+bx}$.

There will be no points of inflection when $f''(x) \geq 0$ for all x.

Solve $9x^4 + 6bx^2 + 6x + b^2 = 0$ and $\dfrac{d}{dx}\left(9x^4 + 6bx^2 + 6x + b^2\right) = 0$

OR

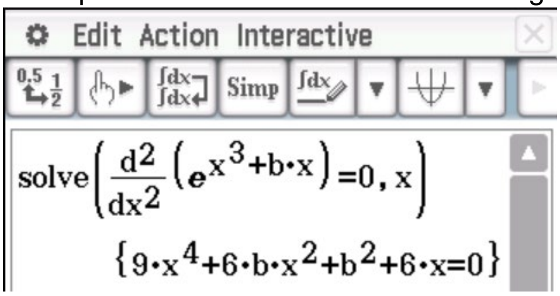
Solve $\left(9b^2x^4 + 6bx^2 + 6bx + 1\right)e^{x^3+bx} = 0$ and $\dfrac{d}{dx}\left(\left(9b^2x^4 + 6bx^2 + 6bx + 1\right)e^{x^3+bx}\right) = 0$

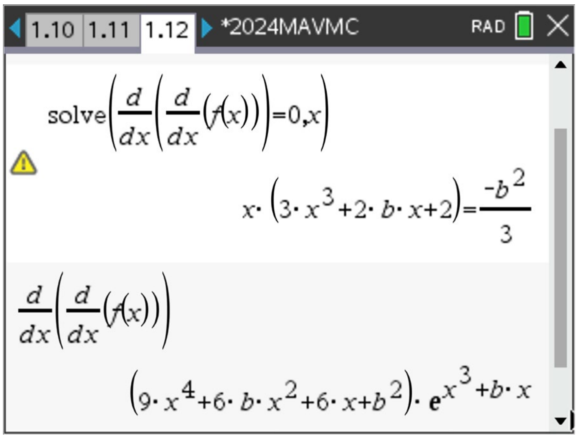
$$b = \frac{3^{\frac{4}{3}}}{4}$$

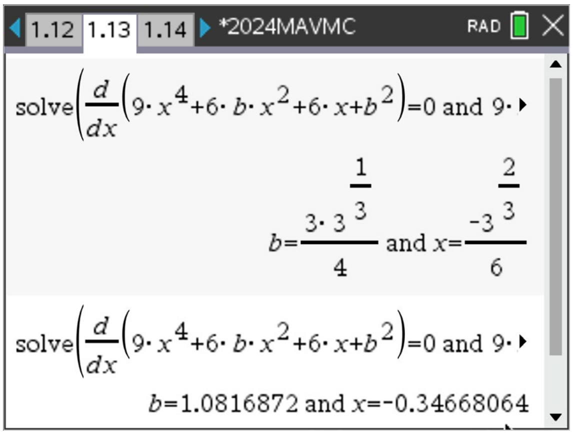
There will be no points of inflection when $b \geq \dfrac{3^{\frac{4}{3}}}{4}$.

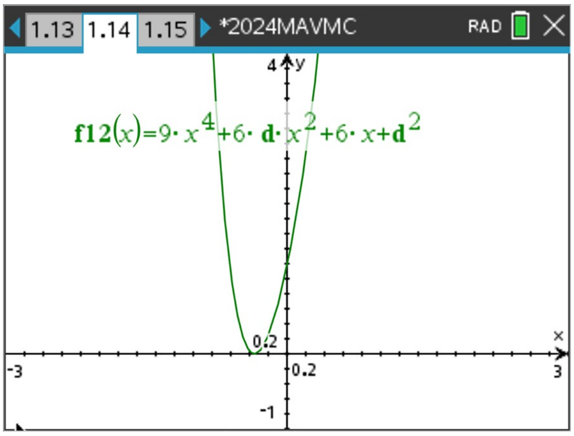
The answer can also be found by checking the values in the options. The easiest way to do this is to graph the function and use a slider. Choose a value of b that gives two points of inflection and label them with their coordinates. Then use the slider to see when they disappear.

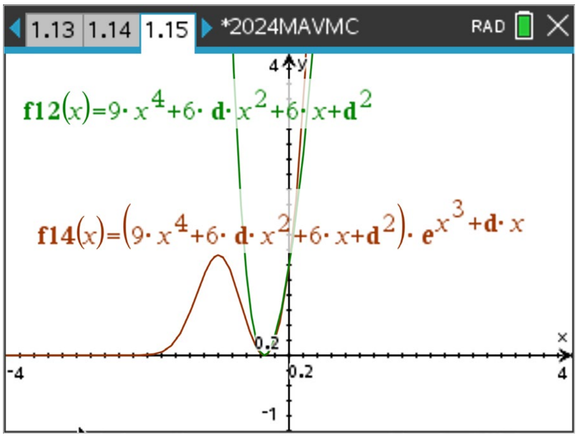
Example of second derivative not working on the CASIO.







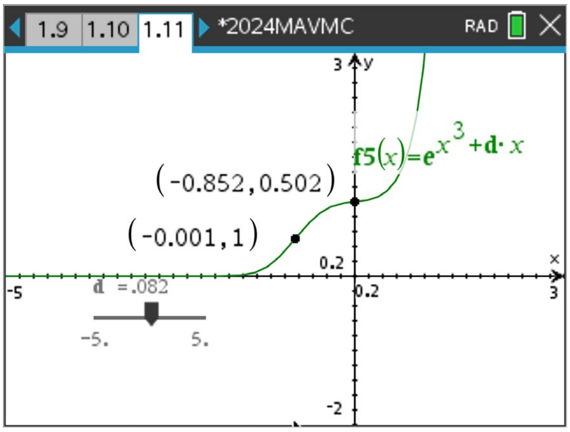


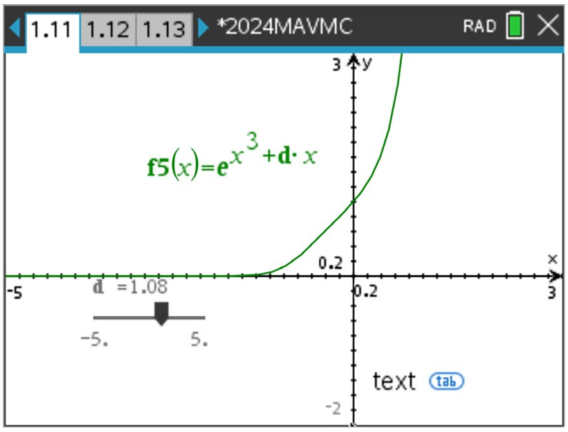


Examples

$b < \frac{3^{\frac{4}{3}}}{4}$ (2 points of inflection)

$b = \frac{3^{\frac{4}{3}}}{4}$ (no points of inflection)





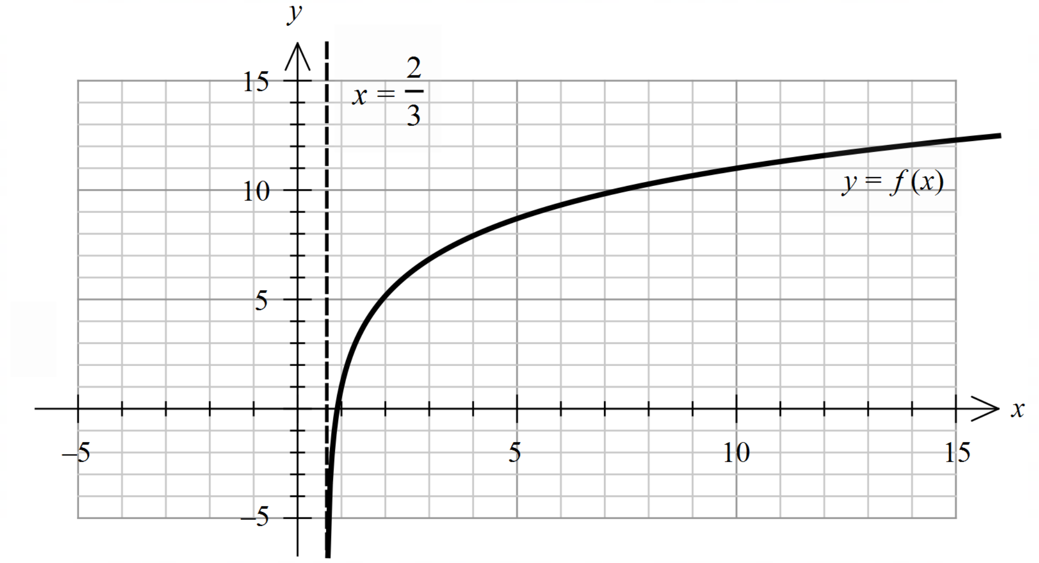
END OF SECTION A SOLUTIONS

SECTION B

Question 1

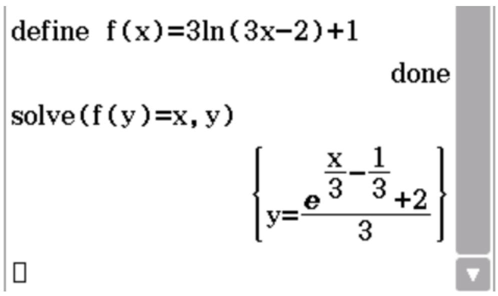
$f:\left(\frac{2}{3},\infty\right)\to R,\ f(x)=3\log_e(3x-2)+1$

a. Sketch and label asymptote $x=\frac{2}{3}$ **1A**



b. $f^{-1}(x)=\frac{1}{3}e^{\frac{x-1}{3}}+\frac{2}{3}$ **1A**

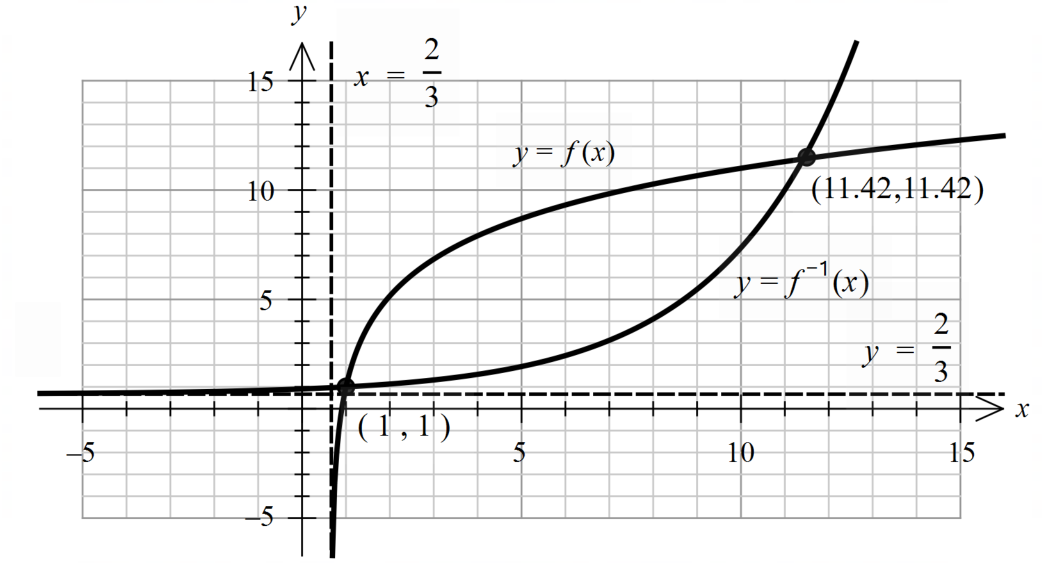
Dom: $x\in R$ **1A**

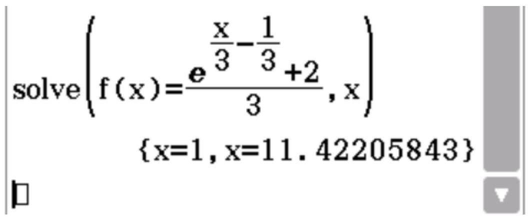


c. Sketch $y=f^{-1}(x)$. Shape and asymptote $y=\frac{2}{3}$ **1A**

Points of intersection between $y=f(x)$ and $y=f^{-1}(x)$

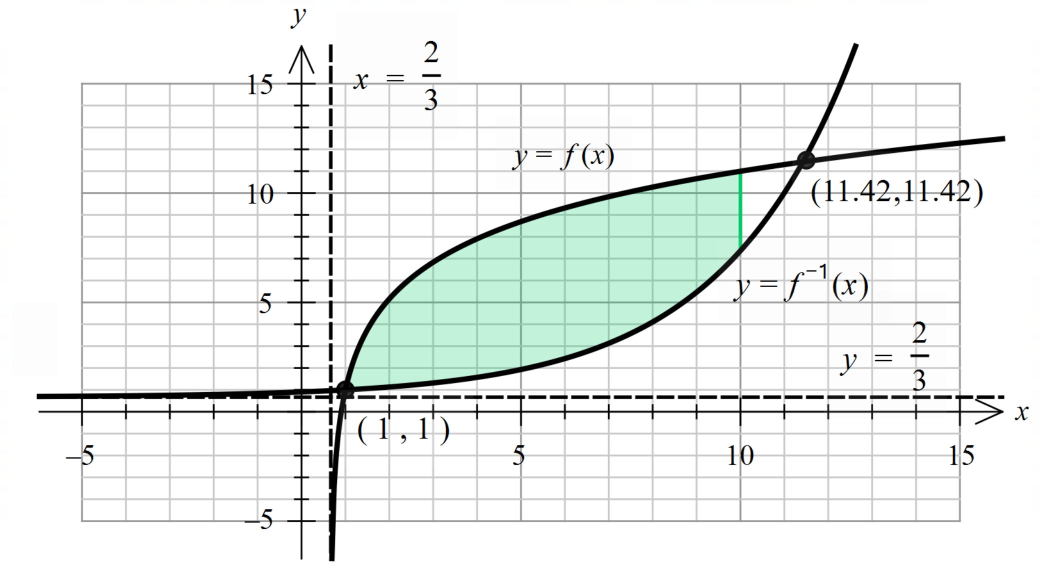
$(1,1)$, $(11.42,11.42)$ **1A**

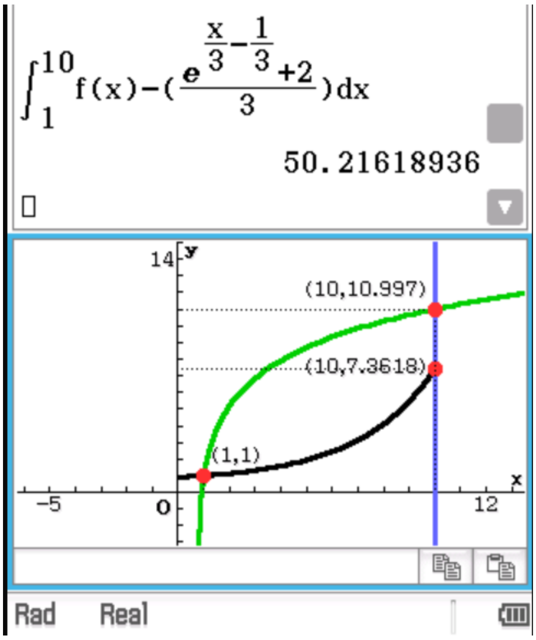


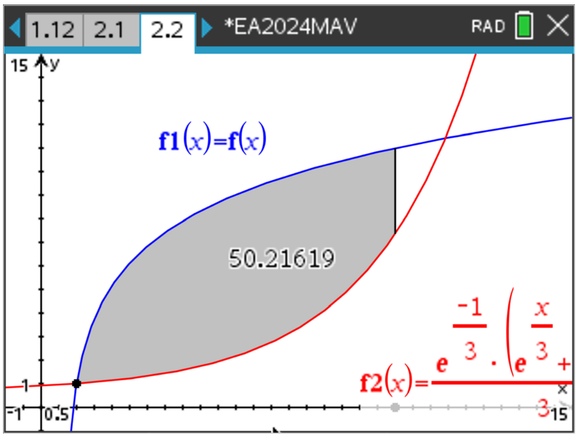


d.i. $\int_{1}^{10}\left(f(x)-f^{-1}(x)\right)dx$ **1A**

d.ii. Area $= 50.22$ sq units **1A**
Shading **1A**

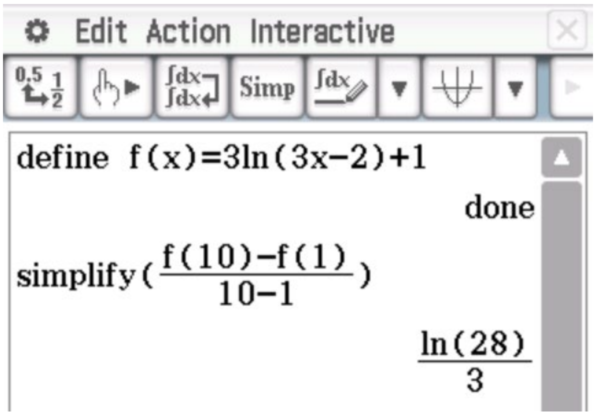






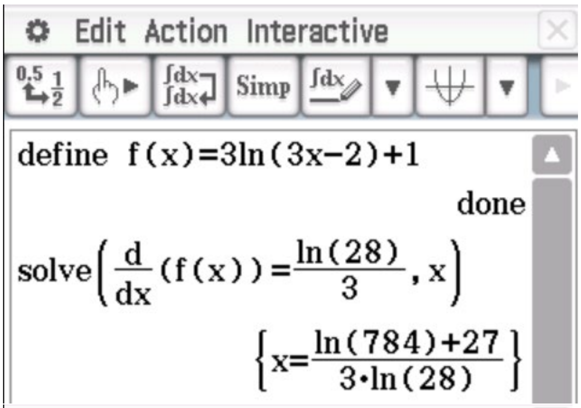
e.i. Average rate of change $= \dfrac{f(10)-f(1)}{10-1}$

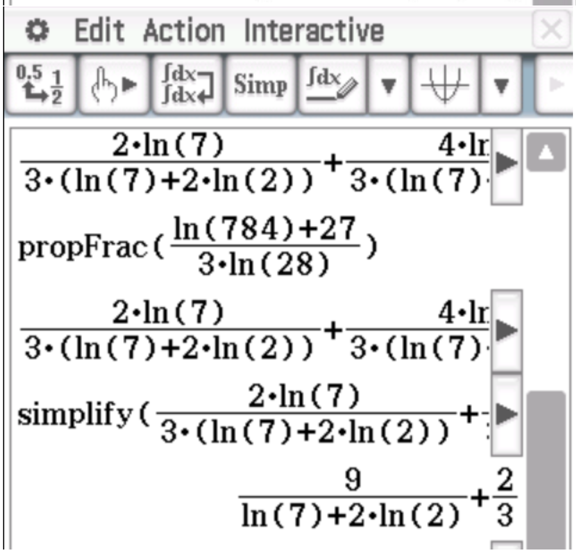
$= \dfrac{\log_e(28)}{3}$. In correct form $\dfrac{\log_e(a)}{b}$ **1A**

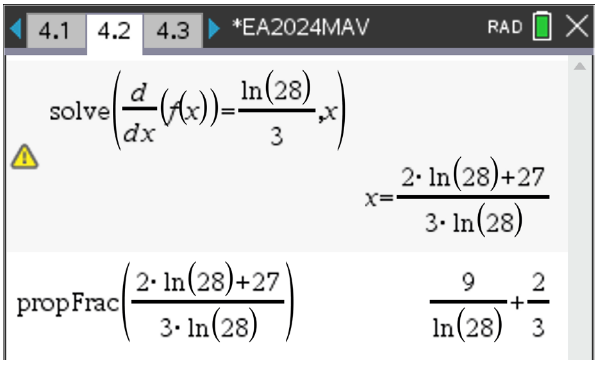


e.ii. Solve $f'(x) = \dfrac{\log_e(28)}{3}$ for x.

$x = \dfrac{2}{3} + \dfrac{9}{\log_e(28)}$ **1A** (other forms)





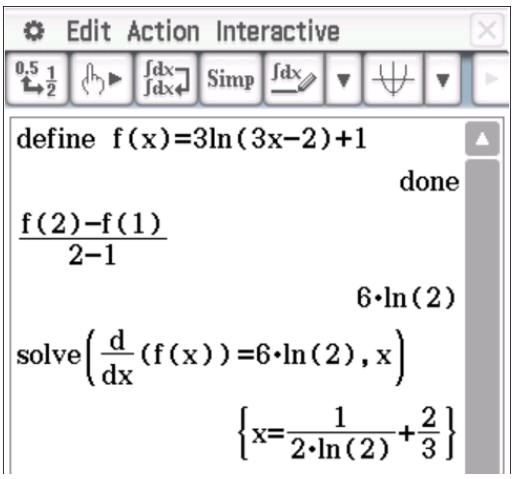


e.iii. The maximum value of the average rate of change will occur when the gradient of the line passing through $(a, f(a))$ and $(b, f(b))$ is steepest. This will occur when $a=1$ and $b=2$.

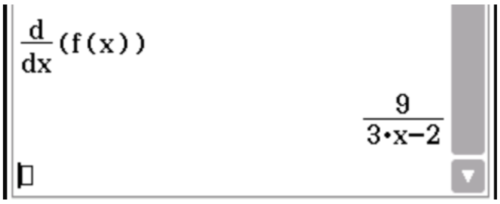
Maximum average rate of change $=\dfrac{f(2)-f(1)}{2-1}=6\log_e(2)$ **1M**

Solve $f'(x)=6\log_e(2)$ for x.

$x=\dfrac{2}{3}+\dfrac{1}{2\log_e(2)}$ **1A** (other forms)



e.iv. f is continuous over the interval $[a,b]$ and smooth over the interval (a,b) but $f'(x)=\dfrac{9}{3x-2}\neq 0$ for any x. Hence, $f(a)\neq f(b)$. For the average value to equal zero, $f(a)$ must equal $f(b)$. **1A**



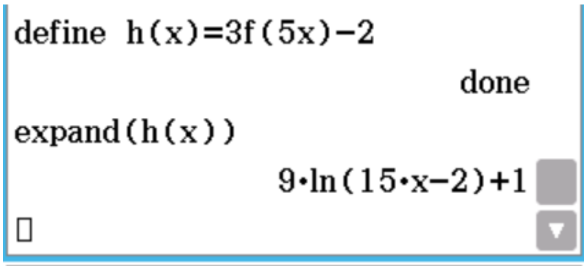
f.i. $h:\left(\dfrac{2}{b},\infty\right)\to R,\ h(x)=a\log_e(bx-2)+1$ where $h(x)=3f(5x)-2$

given $f(x)=3\log_e(3x-2)+1$

$h(x)=3f(5x)-2$

$h(x)=9\log_e(15x-2)+1$

$a=9,\ b=15$ **1A**



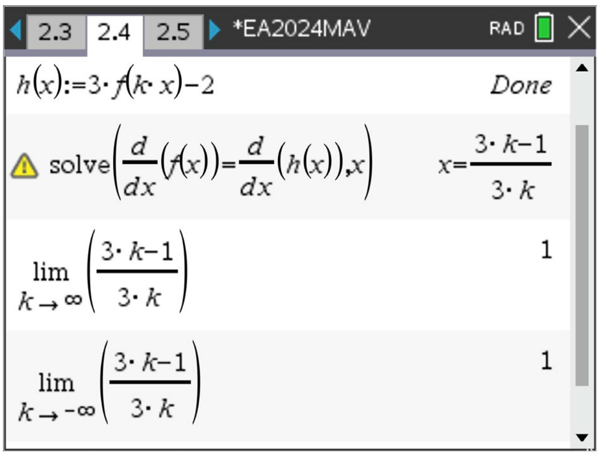
f.ii. Solve $h_1'(x)=f'(x)$

$x=\dfrac{3k-1}{3k}$

As $k\to\infty$, $x\to 1$

As $k\to-\infty$, $x\to 1$

$x=1$ **1A**



Question 2

$h(t) = a\sin\left(b(t-18)\right) + c$

a. max 50, min 10, amp $= 20$

translation vertically: $-20 + 30 = 10$

$a = 20,\ c = 30$ **1M** (explanation)

b. One cycle $= 18$ hours

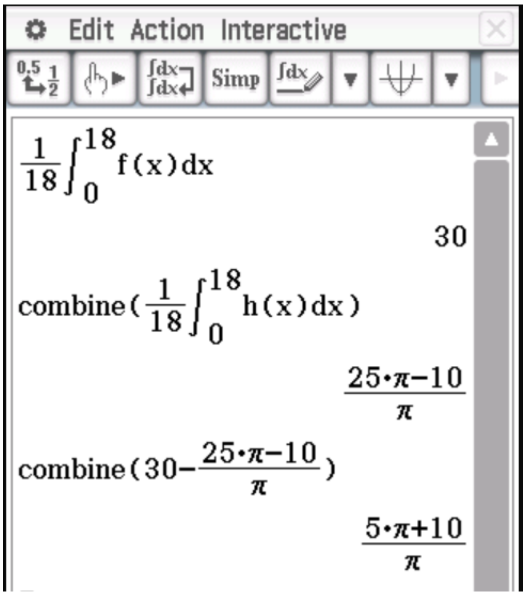
$\frac{2\pi}{b} = 18$

$2\pi = 18b$

Gives $b = \frac{\pi}{9}$ **1M** (show that)

c. $\frac{1}{18}\int_0^{18} h_A dx - \frac{1}{18}\int_0^{18} h_B dx$ **1M**

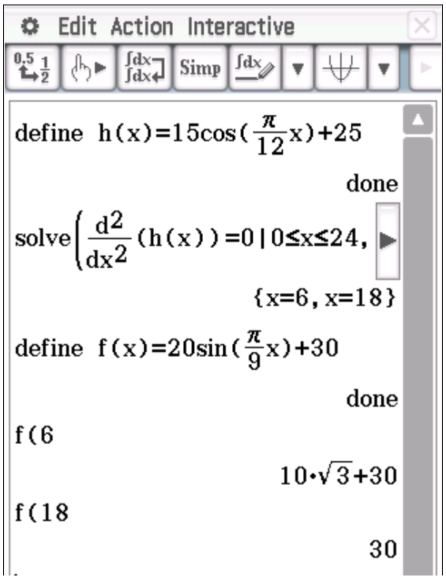
$= \frac{5(\pi+2)}{\pi}$ m **1A**

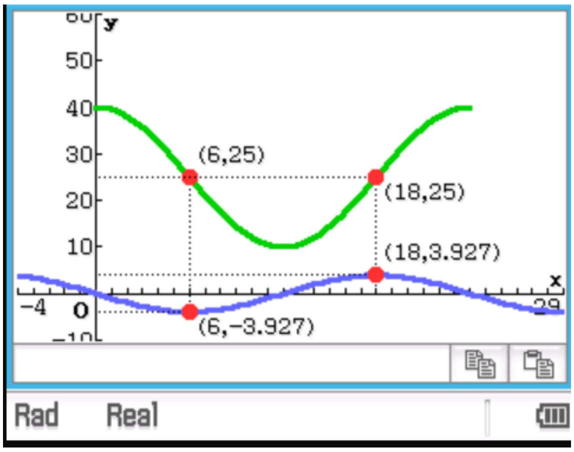


d. $h_B(t) = 15\cos\left(\frac{\pi t}{12}\right) + 25$

The height of the river would be changing fastest at the points of inflection of the graphs of h_B. So when $t = 6$ and $t = 18$. **1M**

$h_A(6) = 10\sqrt{3} + 30$ and $h_A(18) = 30$ **1A**

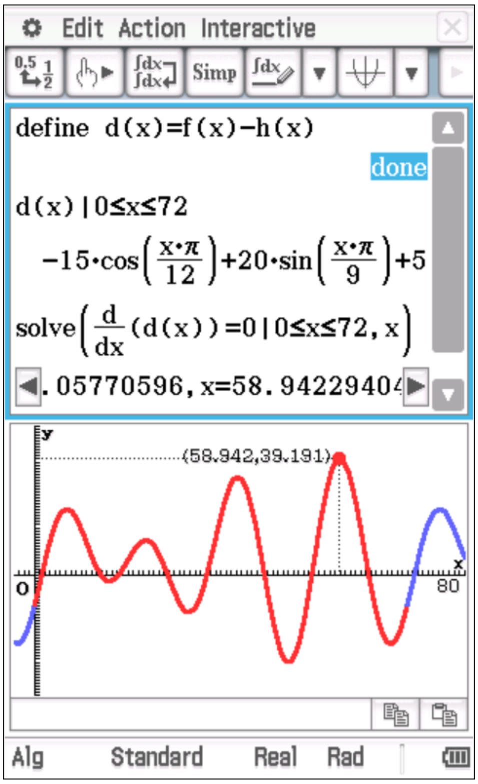


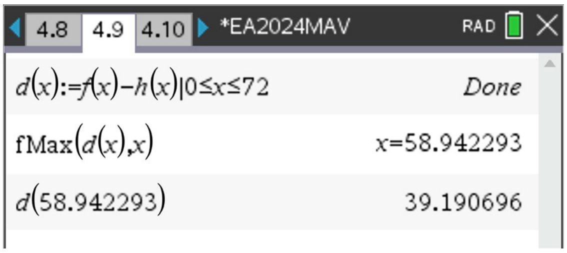


e. Let $d(x) = h_A(x) - h_B(x)$

The period of the graph of $d(x)$ is the lowest common multiple of 18 and 24 which is 72 hours. **1A**

The maximum difference is 39.19 m. **1A**

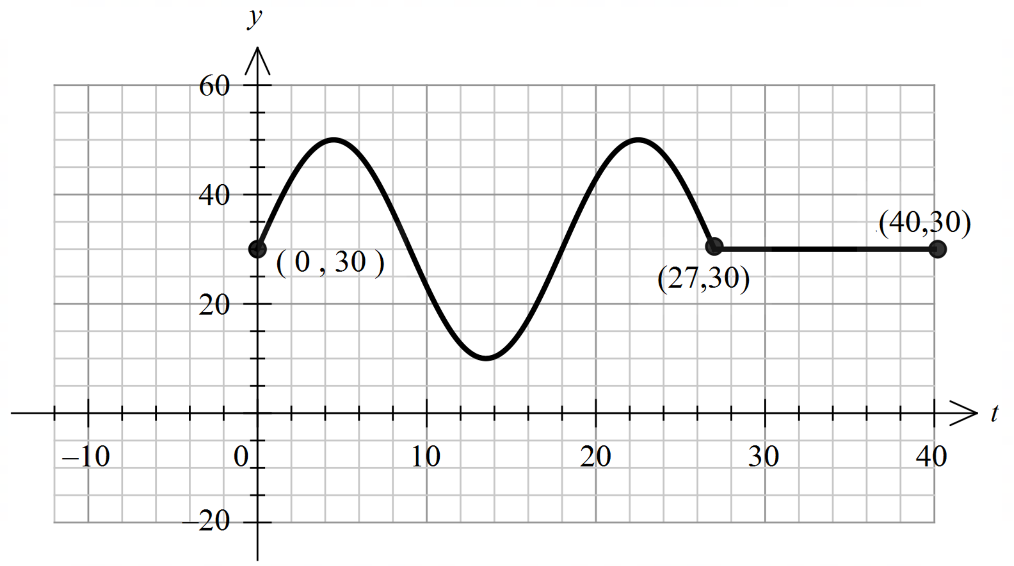


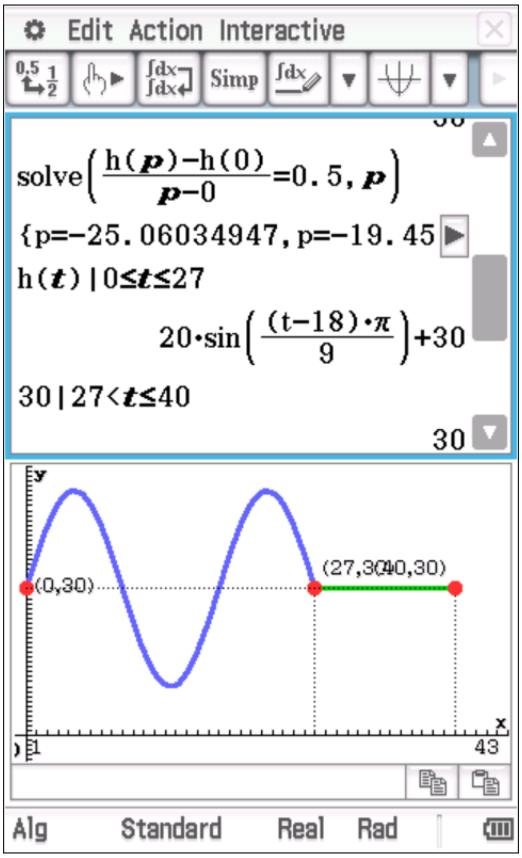


f. $w(t) = \begin{cases} h_A(t) & 0 \le t \le 27 \\ 30 & 27 < t \le 40 \end{cases}$

Graph of piecewise function w **1A**

Coordinates $(0,30)$, $(40,30)$, $(27,30)$ **1A**





g. $t \in (0,27) \cup (27,40)$ **1A**

h. $p(t) = \begin{cases} w(t) & 0 \le t \le 40 \\ m\cos\left(n(t-r)\right) + s & 40 < t \le k \end{cases}$

9 am Sunday to 9 pm Tuesday is 60 hours.

$k = 60$ **1A**

i. Continuous and smooth at $t = 40$. So there is a turning point at $t = 40$.

$p = m\cos(n(t-r)) + s$ completes two cycles before recording capacities break, reaching zero height twice. So the range is $[0,30]$.

Amplitude $=15$, $m=15$

Period $=10$, $n = \frac{2\pi}{10} = \frac{\pi}{5}$ **1H**

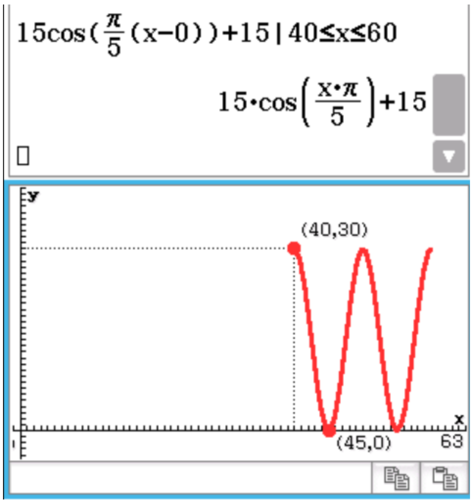
$m=15,\ s=15$. **1A**

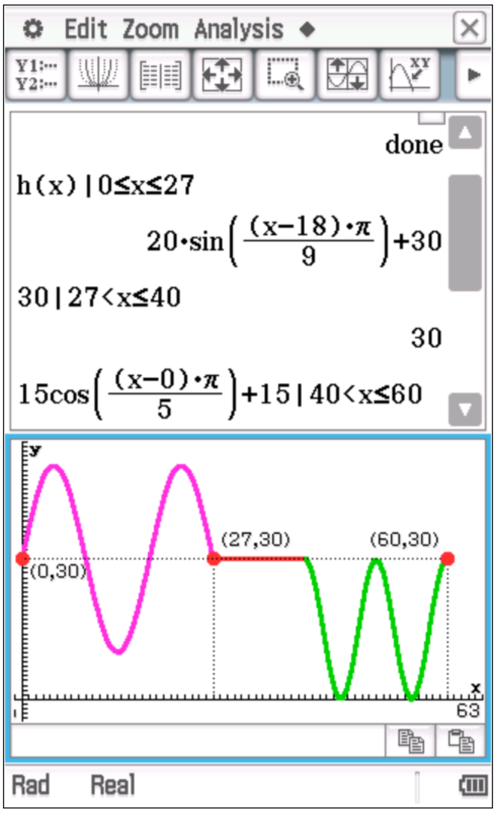
$r=10q$ where $q \in Z$ **1A**

OR

$m=-15,\ s=15$. **1A**

$r=5q$ where $q \in Z$ **1A**



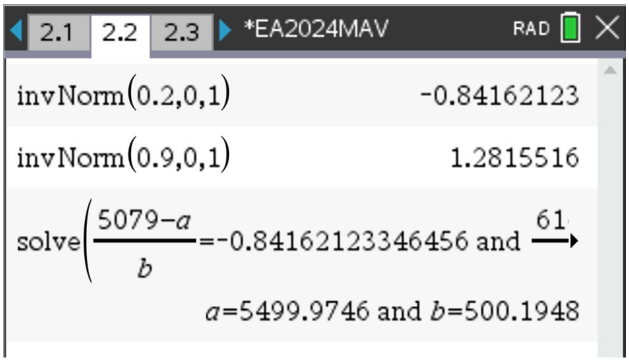


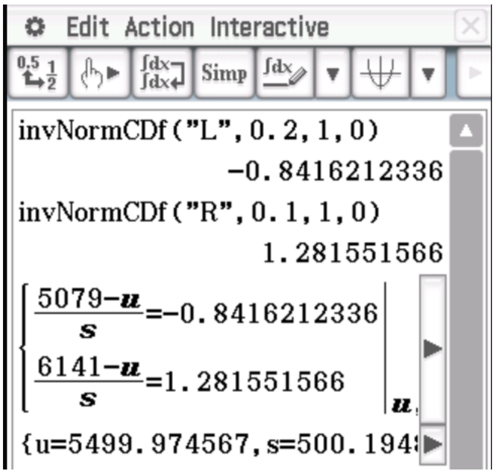
Question 3

a. $X_{AF} \sim \mathrm{N}(\mu, \sigma^2)$

Solve $\frac{5079-\mu}{\sigma} = -0.841...$ and $\frac{6141-\mu}{\sigma} = 1.281...$ **1M**

$\mu = 5500.0$ kg and $\sigma = 500.2$ kg **1A**





b. $X_A \sim \mathrm{N}\left(4085, 445^2\right)$, $X_{AB} \sim \mathrm{N}\left(5375, 225^2\right)$

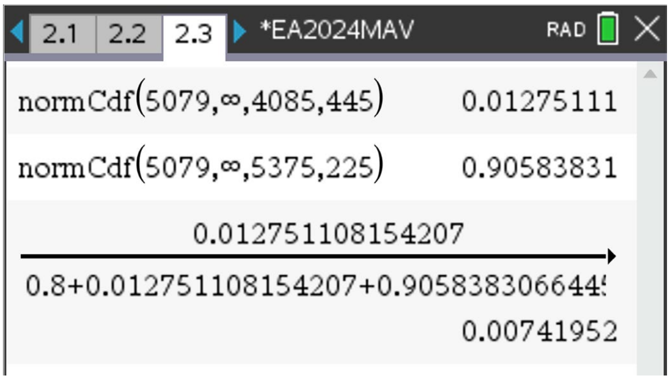
$\Pr\left(X_A > 5079\right) = 0.0127...$, $\Pr\left(X_{AB} > 5079\right) = 0.9058...$, $\Pr\left(X_{AF} > 5079\right) = 0.8$ **1M**

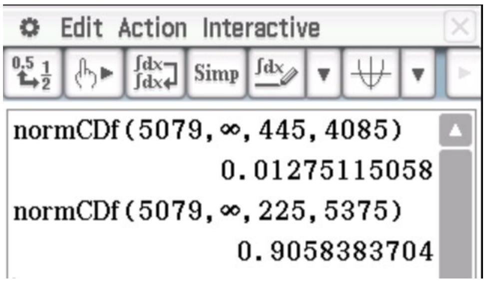
$\Pr\left(X_A > 5079 \mid \left(X_A > 5079 + X_{AB} > 5079 + X_{AF} > 5079\right)\right)$

$$= \frac{\frac{1}{3} \times 0.0127...}{\frac{1}{3} \times 0.0127... + \frac{1}{3} \times 0.9058... + \frac{1}{3} \times 0.8}$$

$$= \frac{0.0127...}{0.0127... + 0.9058... + 0.8}$$

$= 0.0074$ **1A**



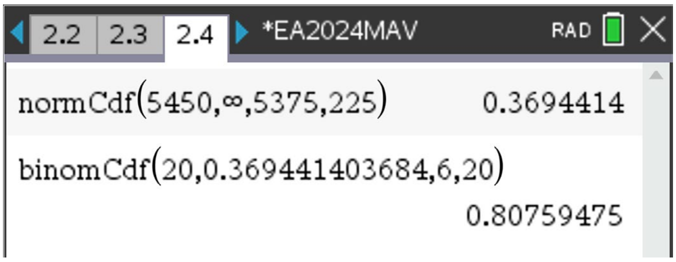


c. $X_{AB} \sim \mathrm{N}\left(5375, 225^2\right)$

$\Pr\left(X_{AB} > 5450\right) = 0.3694...$

$X \sim \mathrm{Bi}\left(20, 0.3694..\right)$ **1M**

$\Pr\left(X > 5\right) = \Pr\left(X \geq 6\right) = 0.8076$ **1A**

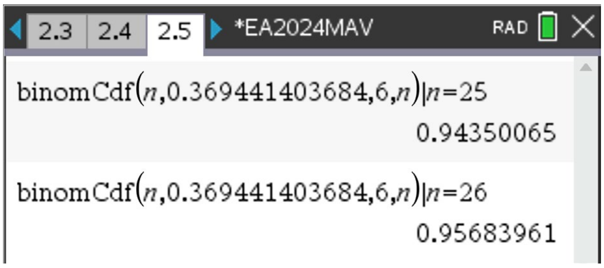


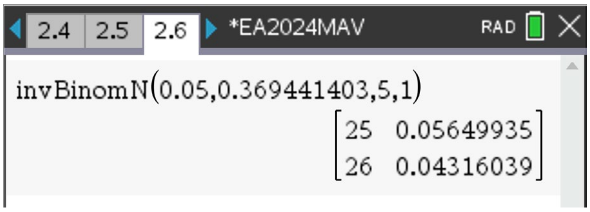
d. $X_2 \sim \text{Bi}(n, 0.3694..)$

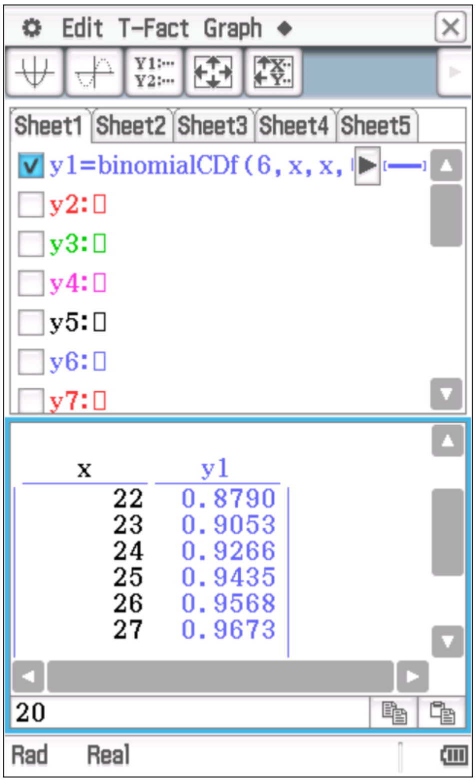
Trial and error **1M** (other methods)

n	$\Pr(X_2 \geq 6)$
25	0.9435...
26	0.9568...

$n = 26$ **1A**

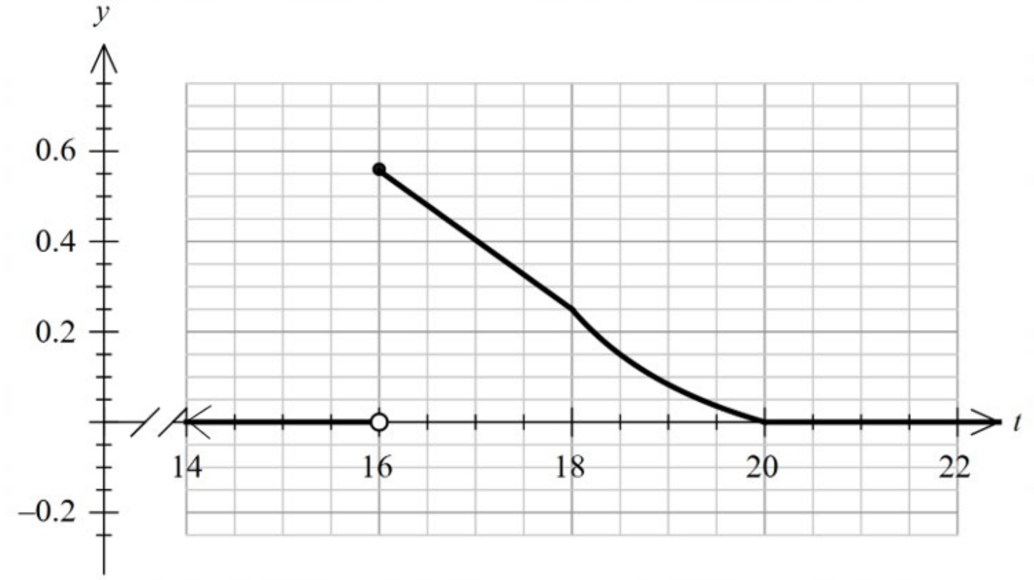






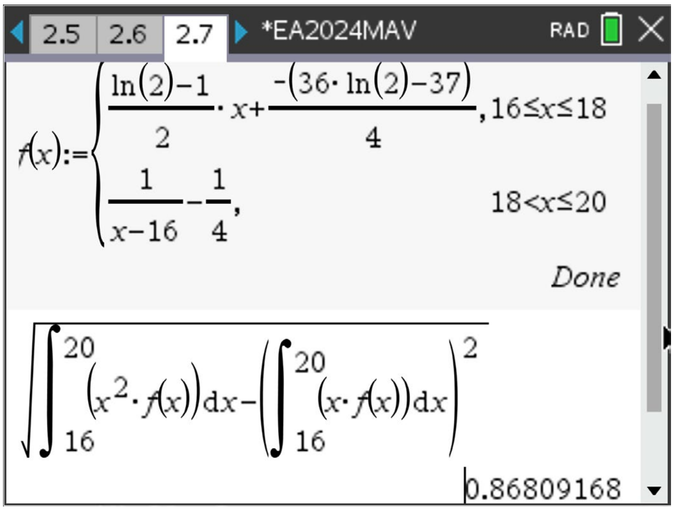
e. Shape, open circle, must draw along the x-axis **1A**

$$f(t)=\begin{cases}\dfrac{\log_e(2)-1}{2}t+\dfrac{37-36\log_e(2)}{4} & 16\le t\le 18\\ \dfrac{1}{t-16}-\dfrac{1}{4} & 18<t\le 20\\ 0 & \text{elsewhere}\end{cases}$$

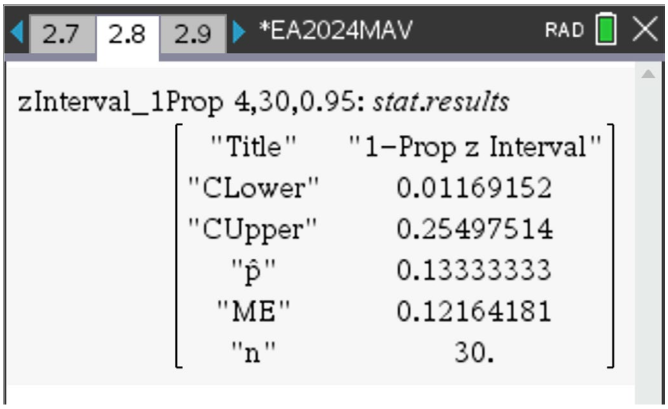


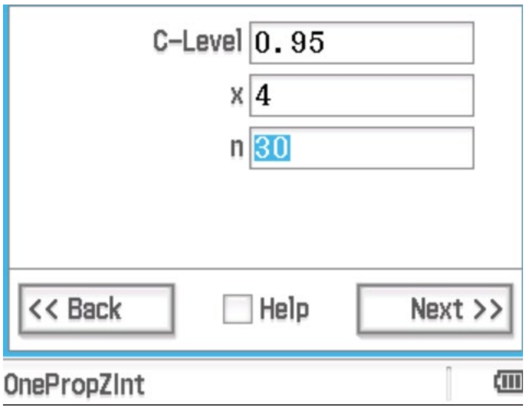
f. $\text{sd}(T)=\sqrt{\int_{16}^{20}\left(t^2\times f(t)\right)dt-\left(\int_{16}^{20}\left(t\times f(t)\right)dt\right)^2}$ **1M**

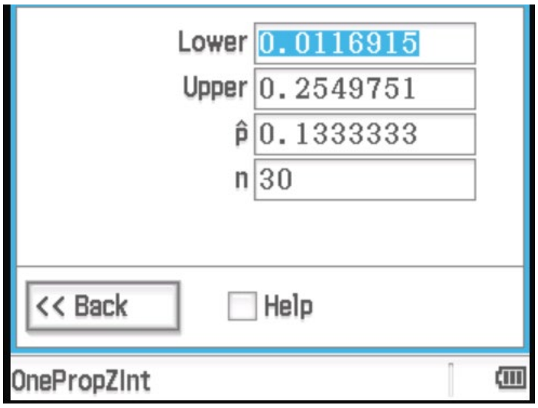
$=0.868$ **1A**



g. $(0.0117, 0.2550)$ **1A**

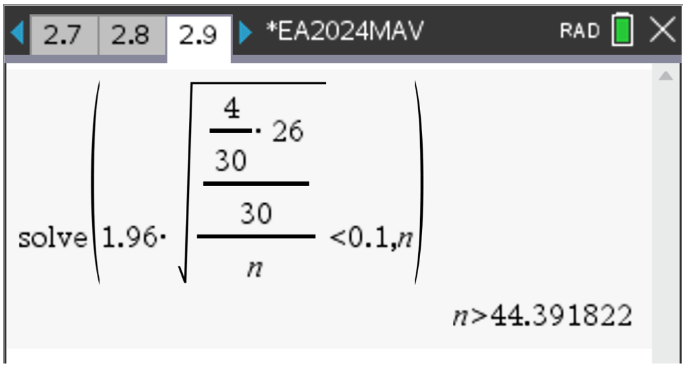






h. Solve $1.96\sqrt{\dfrac{\frac{4}{30}\times\frac{26}{30}}{n}}<0.1$ for n.

$n=45$ **1A**



i. Let AB be the African bush elephant and AF be the African forest elephant.

$\Pr(AB\cap AF)=\Pr(AB)\times\Pr(AF)=\dfrac{2-k^2}{2}$ independent events

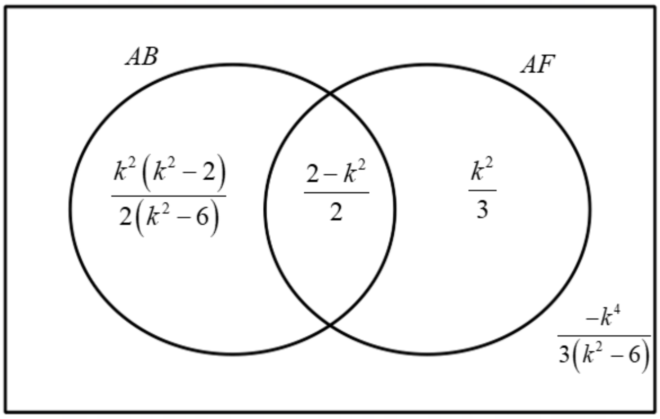
$\Pr(AB)\times\left(\dfrac{2-k^2}{2}+\dfrac{k^2}{3}\right)=\dfrac{2-k^2}{2}$ **1M**

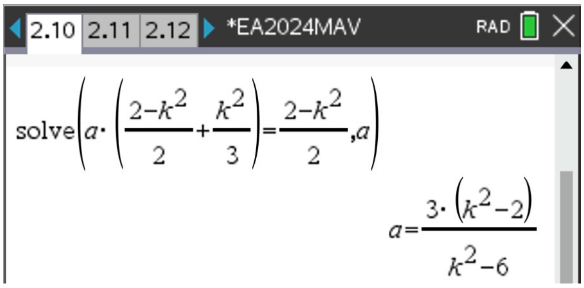
$$\Pr(AB)=\frac{3(k^2-2)}{k^2-6}$$

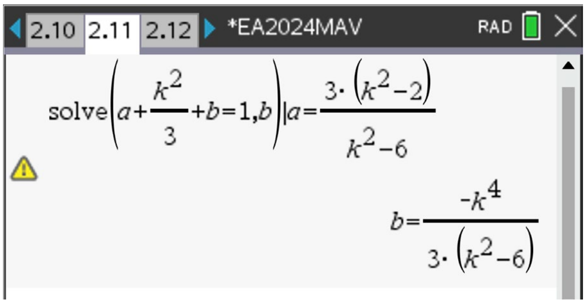
$$\Pr(AB\cup AF)+\Pr(AB'\cap AF')=1$$

$$\frac{3(k^2-2)}{k^2-6}+\frac{k^2}{3}+\Pr(AB'\cap AF')=1$$

$$\Pr(AB'\cap AF')=\frac{-k^4}{3(k^2-6)}$$ **1A**





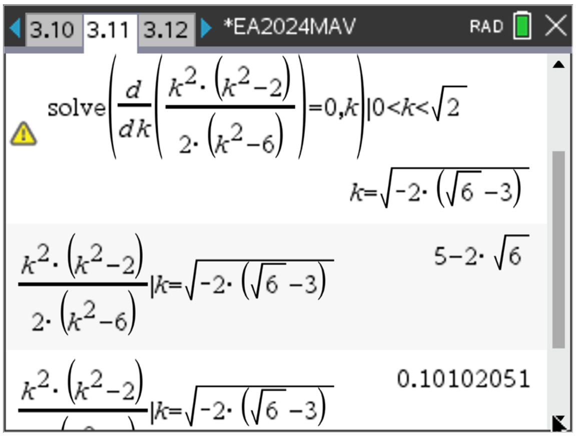


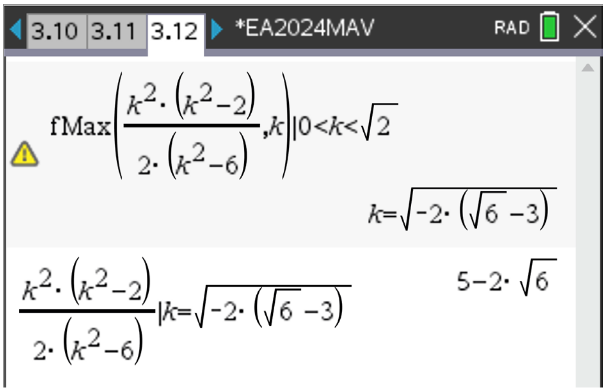
j. $\Pr(AB\cap AF')=\frac{3(k^2-2)}{k^2-6}-\frac{2-k^2}{2}=\frac{k^2(k^2-2)}{2(k^2-6)}$

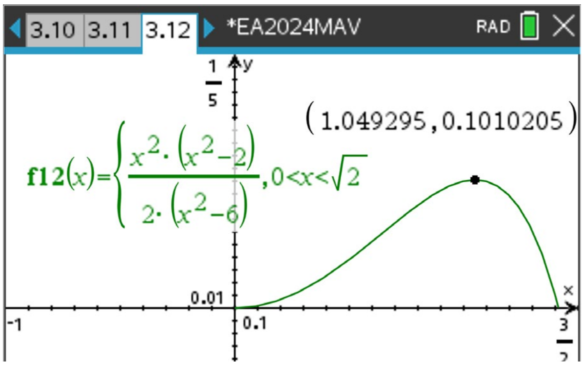
Solve $\frac{d}{dk}\left(\frac{k^2(k^2-2)}{2(k^2-6)}\right)=0$ or use fmax

$k=\sqrt{-2(\sqrt{6}-3)}$

Maximum probability is $5-2\sqrt{6}$ **1A**





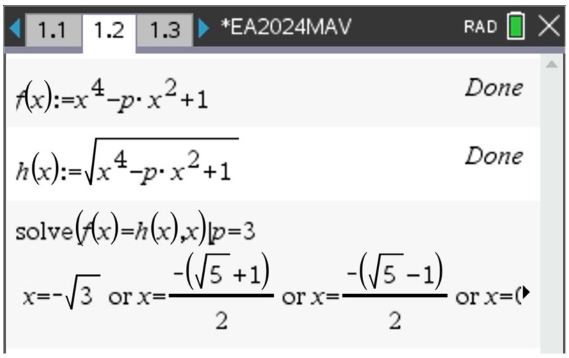


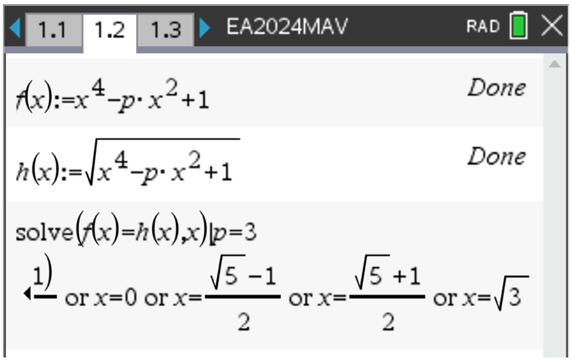
Question 4

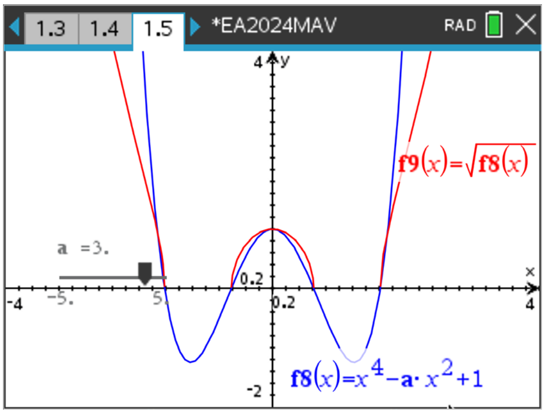
$h(x) = \sqrt{x^4 - px^2 + 1}$ and $f(x) = x^4 - px^2 + 1$, and $p \in R$

a. Solve $h(x) = f(x)$ when $p = 3$

$x = \pm\sqrt{3},\ 0,\ \dfrac{-\sqrt{5} \pm 1}{2},\ \dfrac{\sqrt{5} \pm 1}{2}$ **1A**





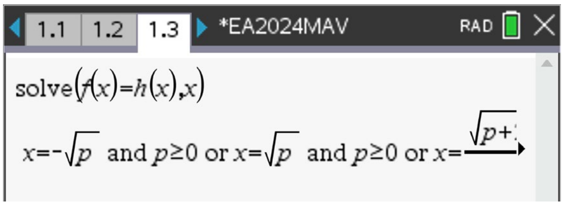


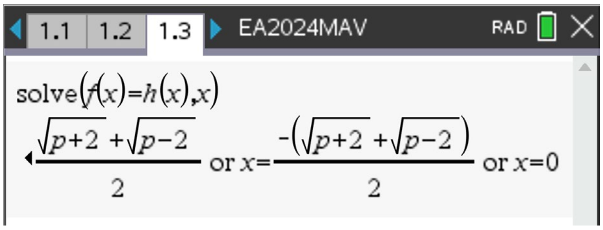
b. Solve $h(x) = f(x)$ for x.

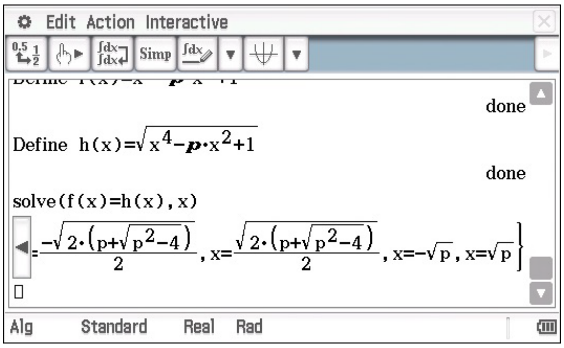
$x = \pm\sqrt{p},\ 0,\ \dfrac{\pm\sqrt{p+2}+\sqrt{p-2}}{2},\ \dfrac{\pm\sqrt{p+2}-\sqrt{p-2}}{2}$ **1A**

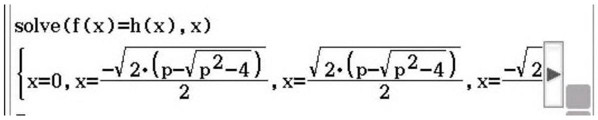
OR

$x = \pm\sqrt{p},\ 0,\ \dfrac{\pm\sqrt{2\left(p+\sqrt{p^2-4}\right)}}{2},\ \dfrac{\pm\sqrt{2\left(p-\sqrt{p^2-4}\right)}}{2}$ **1A**







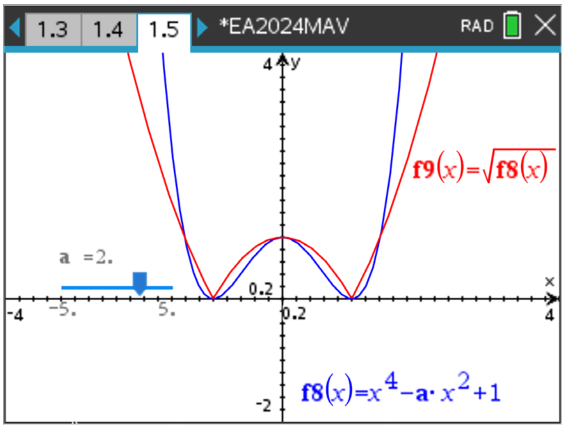


c. If $p = 2$, $\sqrt{p-2} = 0$, so $\dfrac{\pm\sqrt{p+2}+\sqrt{p-2}}{2} = \dfrac{\pm\sqrt{p+2}-\sqrt{p-2}}{2} = \dfrac{\pm\sqrt{p+2}}{2}$

OR

If $p = 2$, $p^2 - 4 = 0$, so $\dfrac{\pm\sqrt{2\left(p+\sqrt{p^2-4}\right)}}{2} = \dfrac{\pm\sqrt{2\left(p-\sqrt{p^2-4}\right)}}{2}$

Hence only five solutions. **1A**

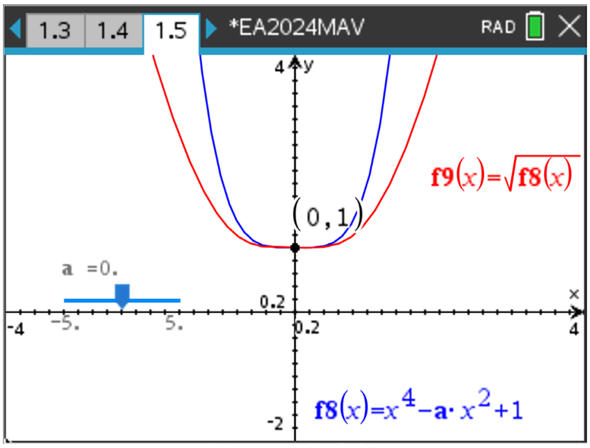


d. 2 correct **1A**
All correct **2A**

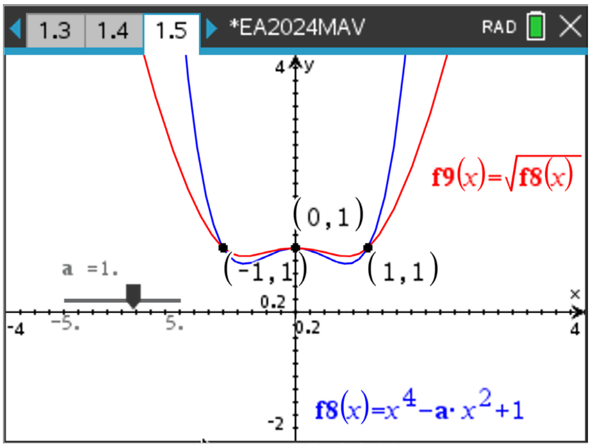
Number of points of intersection	1	3
p values	$p \le 0$	$0 < p < 2$
x-coordinates of points of intersection	0	$0, \ \pm\sqrt{p}$

Examples

$p = 0$



$p = 1$

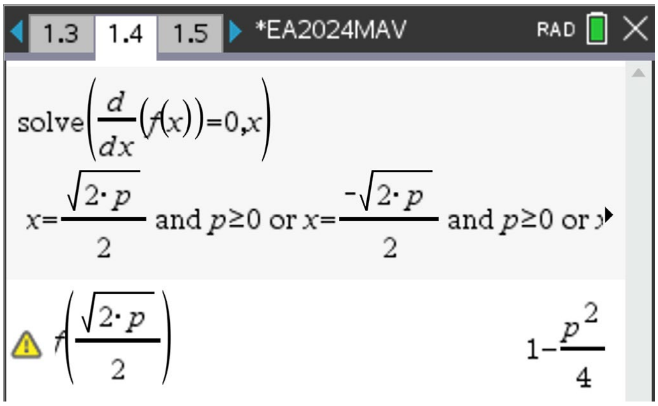


e. $g(x) = \sqrt{x}$, $f(x) = x^4 - px^2 + 1$

For $g\left(f(x)\right)$ to exist the range of f has to be a subset of, or equal to, the domain of g.

The domain of g is $[0, \infty)$, the range of f will be $[0, \infty)$ for $p = 2$ and a subset of $[0, \infty)$ for $p < 2$.

The range of f, $\left[1 - \frac{p^2}{4}, \infty\right)$, is not a subset of $[0, \infty)$ for $p > 2$. **1A**



f. Using the bounded area on the graph

Area = $2(0.0196238...+0.076381...)$ **1M**

= 0.192 **1A**

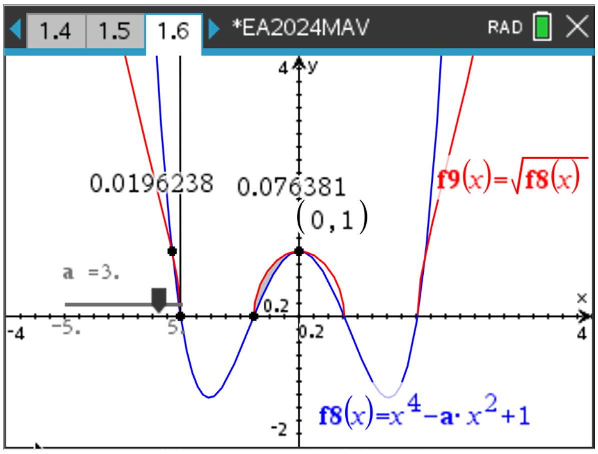
OR

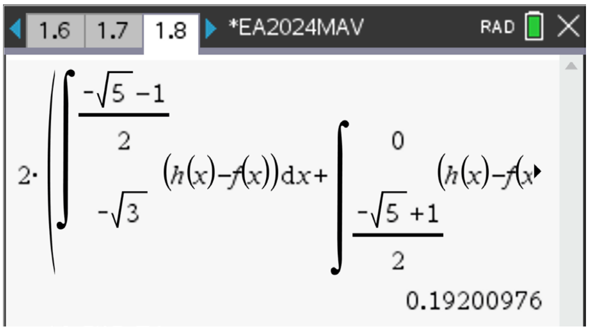
Using definite integrals

Area $= 2\left(\int_{-\sqrt{3}}^{\frac{-\sqrt{5}-1}{2}} \left(h(x)-f(x)\right)dx + \int_{\frac{-\sqrt{5}+1}{2}}^{0} \left(h(x)-f(x)\right)dx\right)$ **OR**

$= 2\left(\int_{0}^{\frac{\sqrt{5}-1}{2}} \left(h(x)-f(x)\right)dx + \int_{\frac{\sqrt{5}+1}{2}}^{\sqrt{3}} \left(h(x)-f(x)\right)dx\right)$ **1M** (either form)

$= 0.192$ **1A**





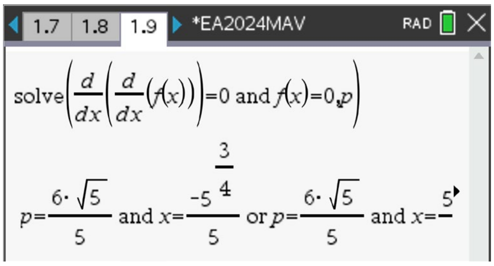
g. Area $= 2\left(\int_{-\sqrt{p}}^{\frac{-\sqrt{p+2}-\sqrt{p-2}}{2}} \left(h(x)-f(x)\right)dx + \int_{\frac{-\sqrt{p+2}+\sqrt{p-2}}{2}}^{0} \left(h(x)-f(x)\right)dx\right)$ **OR**

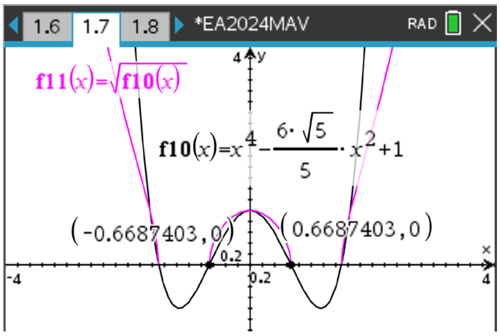
$= 2\left(\int_{0}^{\frac{\sqrt{p+2}-\sqrt{p-2}}{2}} \left(h(x)-f(x)\right)dx + \int_{\frac{\sqrt{p+2}+\sqrt{p-2}}{2}}^{\sqrt{p}} \left(h(x)-f(x)\right)dx\right)$ **OR**

$$=2\left(\int_{0}^{\frac{\sqrt{2\left(p-\sqrt{p^2-4}\right)}}{2}}\left(h(x)-f(x)\right)dx+\int_{\frac{\sqrt{2\left(p+\sqrt{p^2-4}\right)}}{2}}^{\sqrt{p}}\left(h(x)-f(x)\right)dx\right)$$ **1A**

h. Solve $f''(x)=0$ and $f(x)=0$ for p. **1M**

$p=\frac{6\sqrt{5}}{5}$ **1A**





i. $h_v(x)=\sqrt{x^4-3x^2+1}$ and $f_v(x)=x^4-3x^2+1$

Cross-sectional area $=\int_{-1.817...}^{1.817...}\left(2-f_v(x)\right)dx-0.1920...=7.517...$ **1M**

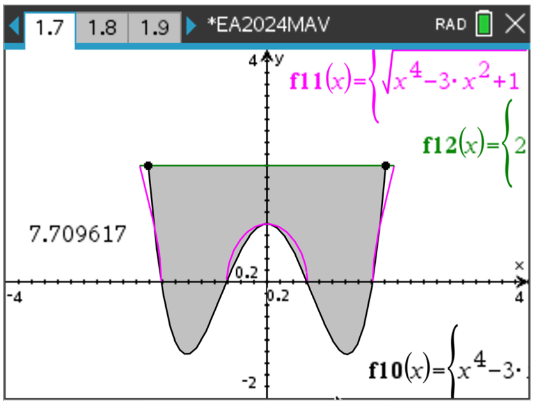
Volume = $=7.517...\times 3=22.552...$

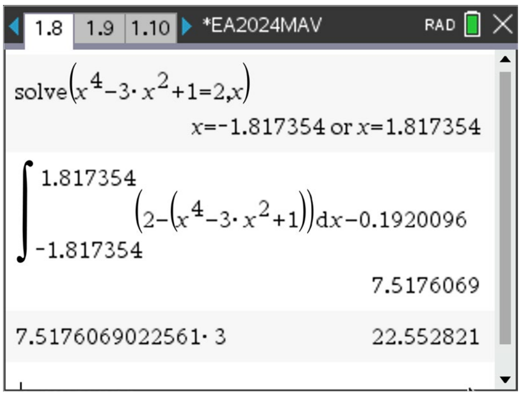
$\frac{dv}{dt}=2^t$

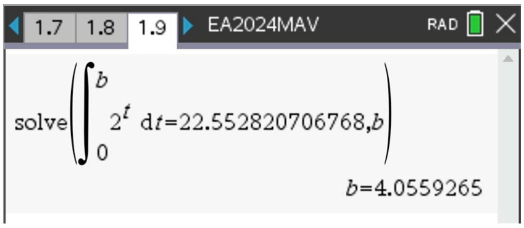
Solve $\int_0^b 2^t\,dt=22.552...$ for b. **1H**

$b=4.06$ seconds **1A**

The shaded area on the graph is $\int_{-1.817...}^{1.817...}\left(2-f_v(x)\right)dx=7.709...$ and then you need to subtract the bound area found in **part f**.







END OF SOLUTIONS