The Mathematical Association of Victoria

Trial Examination 2024

MATHEMATICAL METHODS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of examination

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	4	4	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 20 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions

- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination

- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A- Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple – choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The amplitude, A, and the period, P, of the function $f(x) = -\frac{3}{2}\sin\left(2x - \pi\right)$ are

A. $A = -\frac{3}{2},\ P = \pi$

B. $A = -\frac{3}{2},\ P = \frac{\pi}{2}$

C. $A = \frac{3}{2},\ P = \pi$

D. $A = \frac{3}{2},\ P = 2$

Question 2

If $f(x) = \sqrt{x+2}$ and $g(x) = e^{2x}$ exist over their maximal domains, then the domain of $g \circ f$ is

A. $[0, \infty)$

B. $[2, \infty)$

C. $[-2, \infty)$

D. $(-\infty, \infty)$

Question 3

The equation $0 = ax^2 + 4x + c$ where $a, c \in R$, will have two unique solutions if

A. $ac < 4$
B. $ac \le 4$
C. $ac > -4$
D. $ac > 4$

Question 4

Consider the system of simultaneous linear equations below, where m is a real constant.

$$x + (m-1)y = 2$$
$$(m+1)x + 3y = 8 - m$$

The system of equations will have an infinite number of solutions if

A. $m = -2$ or $m = 2$
B. $m = 2$
C. $m = -2$
D. $m \in R \setminus \{-2, 2\}$

Question 5

The maximal interval over which the graph of $g(x)=x^3+\frac{9}{8}x^2-\frac{3}{4}x$ is strictly decreasing is

A. $\left(-1,\frac{1}{4}\right)$

B. $\left[-1,\frac{1}{4}\right]$

C. $(-\infty,-1]\cup\left[\frac{1}{4},\infty\right)$

D. $(-\infty,-1]\cup[1,\infty)$

Question 6

Which one of the following functions is differentiable over its maximal domain?

A. $y=x^{\frac{1}{3}}$

B. $y=x^{\frac{2}{3}}$

C. $y=x^{\frac{1}{2}}$

D. $y=x^{\frac{4}{3}}$

Question 7

Suppose that $\int_1^3 f(x)dx=4$ and $\int_3^1 g(x)dx=-2$. The value of $-\int_1^2 g(x)dx+\int_1^3 (2f(x)+3)dx-\int_2^3 g(x)dx$ is

A. 9
B. 12
C. 13
D. 16

Question 8

Two boxes, box A and box B each contain 4 red balls and 3 green balls. A box is randomly selected and from that box a ball is selected at random and not replaced, then another ball is selected at random from that same box. The probability that two balls of the same colour are selected is

A. $\frac{25}{98}$

B. $\frac{25}{49}$

C. $\frac{3}{14}$

D. $\frac{3}{7}$

Question 9

Let $h: R \setminus \{1\} \to R,\ h(x) = \frac{1}{x-1} + 2$.

The average rate of change of h from $x = 2$ to $x = 5$ is equal to the instantaneous rate of change of h when

A. $x = -1$ only
B. $x = 3$ only
C. $x = -1$ or $x = 3$
D. $x = 2$

Question 10

A continuous random variable X has the following probability density function.

$$f(x) = \begin{cases} k\sin\left(\frac{1}{2}x\right) & 0 < x < \pi \\ 0 & \text{otherwise} \end{cases} \quad \text{where } k \in R$$

The value of m such that $\Pr(X < m) = 0.5$ is

A. $\frac{\pi}{3}$
B. $\frac{2}{3}$
C. 1.44
D. $\frac{2\pi}{3}$

Question 11

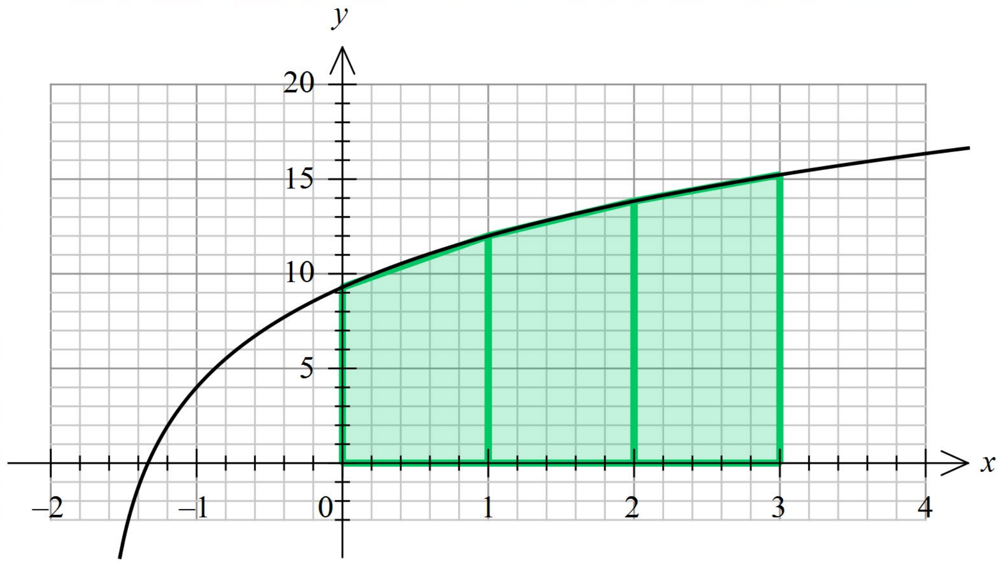
Let $f: R \setminus \{1\} \to R, f(x) = \frac{1}{(x-1)^2} - 2$.

The perimeter of the triangle formed by the tangent lines to the graph of f at $x = 0$ and $x = 2$ and the line $y = -2$ is

A. $3\sqrt{5} + 3$
B. 9
C. $\frac{9}{2}$
D. $3\sqrt{3} + 3$

Question 12

The area between the graph of $y = g(x) = 4\log_2(3x+5)$, the x-axis and the lines $x = 0$ and $x = 3$ has been approximated using three trapeziums of equal length, as shown in the graph below.



This approximate area is **not** equal to

A. $2\left(\log_2(5)+\log_2(14)+\log_2(64)+\log_2(121)\right)$

B. $4\left(\log_2\left(\sqrt{5}\right)+3+\log_2(11)+\log_2\left(\sqrt{14}\right)\right)$

C. $\dfrac{2}{\log_e(2)}\left(\log_e(5)+2\log_e(8)+2\log_e(11)+\log_e(14)\right)$

D. $\log_2(5\times 14\times 121)^2+6^2$

Question 13

A general solution for the equations of the asymptotes of the graph of $f(x) = 3\tan\left(\frac{1}{2}\left(\frac{\pi}{3}x-1\right)\right)+5$ over its maximal domain is

A. $y = \dfrac{3(\pi+1)}{\pi}+6k,\ k \in Z$

B. $x = \dfrac{3(\pi+1)}{\pi}+12k,\ k \in Z$

C. $x = \dfrac{-3(\pi-1)}{\pi}+6k,\ k \in Z$

D. $x = \dfrac{-3(\pi-1)}{\pi}+2\pi k,\ k \in Z$

Question 14
The probability mass function for the number of typing errors on a particular Mathematics examination, X, is shown below where a and k are real constants.

x	0	1	2	3
$\Pr(X = x)$	0.2	0.1	a	$\frac{k}{3}$

If the variance is 1.4, then the mean is

A. 2
B. 0.2
C. 1.5
D. 2.1

Question 15
If $s(x) = 1 - \log_e(1-x)$ and $t(x) = 3\cos(2x-1) + 1$ exist over their maximal domains, then the domain of $s(x) + t^{-1}(x)$ is

A. $[-2,4]$
B. $[-2,1)$
C. $(-\infty,1)$
D. $(-\infty,4]$

Question 16
Let $f: R\setminus\left\{\frac{a}{4}\right\} \to R,\ f(x) = \frac{2}{4x-a} + 3$, where a is a real constant.
Newton's method, with an initial estimate x_0, will fail when finding the root for the function f, if

A. $x_0 \in R\setminus\left[\frac{3a-4}{12}, \frac{a}{4}\right]$ only

B. $x_0 = \frac{a}{4}$ only

C. $\frac{3a-4}{12} \le x_0 \le \frac{a}{4}$

D. $x_0 \in R\setminus\left(\frac{3a-4}{12}, \frac{a}{4}\right)$

Question 17
The probability a particular type of calculator will be able to solve a difficult exponential equation is 0.35.
A student had thirty such equations to solve for homework using this particular type of calculator. The probability, correct to four decimal places, that the student was able to solve more than twelve of the equations, given that they were able to solve at least five equations, is

A. 0.2215
B. 0.2198
C. 0.3478
D. 0.3452

Question 18

Let A_v be the average value of $f(x) = x^3 + x^2 - x + 1$ for the interval $[a,1]$, where $a \in (-\infty, 1)$.
There will be three different values of a which will give the same average value when

A. $A_v \in \left[\dfrac{-2\left(20\sqrt{10}-457\right)}{729}, \dfrac{2\left(20\sqrt{10}+457\right)}{729}\right]$

B. $A_v \in [1.08, 1.43]$

C. $A_v \in \left(\dfrac{-2\left(20\sqrt{10}-457\right)}{729}, \dfrac{2\left(20\sqrt{10}+457\right)}{729}\right)$

D. $A_v \in \left(\dfrac{-\left(2\sqrt{10}+7\right)}{9}, \dfrac{2\sqrt{10}-7}{9}\right)$

Question 19

The sequence of transformations that takes the graph of the probability density function with equation

$f(x) = \dfrac{1}{\sqrt{18\pi}} e^{-\frac{1}{2}\left(\frac{2x-3}{6}\right)^2}$ to the graph of the probability density function of the standard normal distribution is

A. a translation of 3 units left, a dilation by a factor of 6 from the x-axis and $\dfrac{1}{6}$ from the y-axis.

B. a dilation by a factor of 3 from the x-axis and $\dfrac{1}{3}$ from the y-axis and a translation of $\dfrac{1}{2}$ a unit left.

C. a translation of $\dfrac{3}{2}$ units right, a dilation by a factor of 3 from the x-axis and $\dfrac{1}{3}$ from the y-axis.

D. a translation of $\dfrac{3}{2}$ units left, a dilation by a factor of $\dfrac{1}{3}$ from the x-axis and 3 from the y-axis.

Question 20

Let $f: R \to R, f(x) = e^{x^3 + bx}$ where b is a real constant.
The graphs of f will have no points of inflection if

A. $b \geq \dfrac{3^{\frac{4}{3}}}{4}$

B. $b > \dfrac{3^{\frac{4}{3}}}{4}$

C. $b > 1.08$

D. $b < \dfrac{3^{\frac{4}{3}}}{4}$

END OF SECTION A
TURN OVER

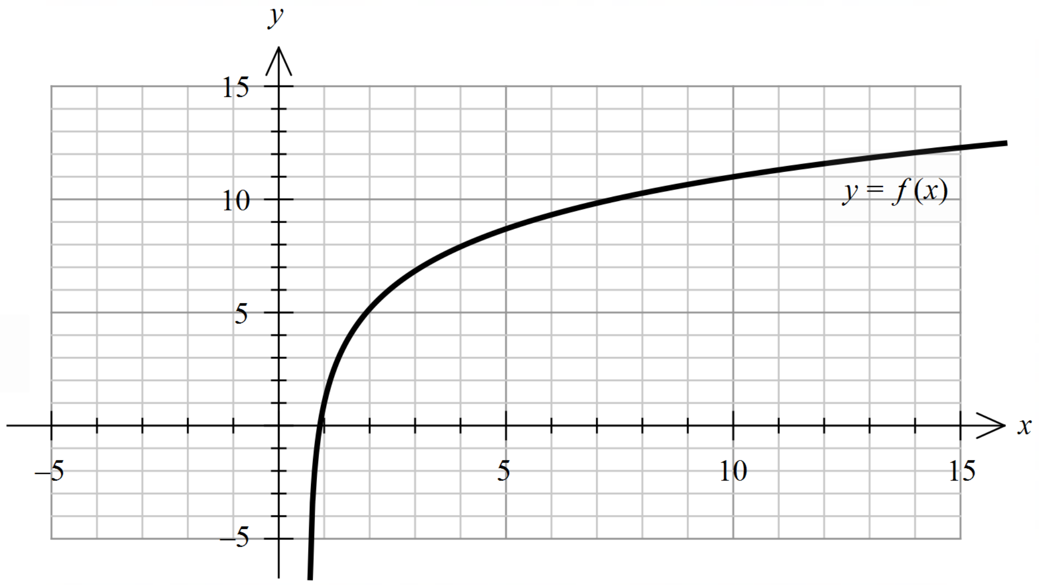
SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (15 marks)

Let $f:\left(\frac{2}{3},\infty\right)\to R, f(x)=3\log_e(3x-2)+1$. Part of the graph of $y=f(x)$ is shown below.



a. On the set of axes above sketch the vertical asymptote and label it with its equation. 1 mark

b. Find the rule and domain of the inverse function, f^{-1}. 2 marks

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c. Sketch the graph of the inverse function $y=f^{-1}(x)$ on the axes above.
Label the points of intersection with their coordinates, giving non-integer values correct to two decimal places. Label the asymptote with its equation. 2 marks

SECTION B – Question 1 – continued

d. i. Write down an expression, using a definite integral, that gives the larger area of the two regions bound by f and f^{-1} and the line $x = 10$. 1 mark

ii. Hence shade the required region on the set of axes on page 8 and find its area, correct to two decimal places. 2 marks

The Mean Value Theorem states that if a function f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), there exists a point c in the interval (a,b) such that $f'(c)$ is equal to the function's average rate of change over $[a,b]$.

e. i. Find the average rate of change of f from $x = 1$ to $x = 10$. Write your answer in the form $\frac{\log_e(m)}{n}$, where $m,\ n \in Z$. 1 mark

ii. Find the x value where the instantaneous rate of change of f equals the average rate of change found in **part e.i**. 1 mark

iii. Find the x value where the instantaneous rate of change of f equals the maximum value of the average rate of change of f from $x = a$ to $x = b$, where $b > a$ and $a,\ b \in Z^+$. 2 marks

Rolle's theorem states that if a function f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) such that $f(a)=f(b)$, then $f'(x)=0$ for some x with $a \le x \le b$.

iv. Use Rolle's theorem to explain why the average rate of change between two points on the graph of f will never equal zero. 1 mark

Let $h:\left(\frac{2}{b},\infty\right) \to R,\ h(x)=a\log_e(bx-2)+1$ where $h(x)=3f(5x)-2$ and $a,b \in R$.

f. **i.** Find the values of a and b. 1 mark

Now consider $h_1:\left(\frac{2}{k},\infty\right) \to R,\ h_1(x)=a\log_e(kx-2)+1$ where $h_1(x)=3f(kx)-2$ and $k \in R$.

ii. What is the limiting value of the solution to $h_1'(x)=f'(x)$ as $k \to \pm\infty$? 1 mark

SECTION B – continued

Question 2 (15 marks)
A tidal river, at a particular station, Station A, has its height above ground level modelled by the function

$$h_A(t) = a\sin\left(b(t-18)\right) + c$$

for some $a, b, c \in R^+$, with height, h_A, in metres t hours after 9 am on a particular day.
On this day the river had a maximum height of 50 metres and minimum height of 10 metres.

a. Explain why $a = 20$ and $c = 30$. 1 mark

One complete cycle of the height, h, of the tidal river is completed in 18 hours.

b. Show that $b = \frac{\pi}{9}$. 1 mark

The height of the river at another station, Station B, further upstream can be modelled by the function

$$h_B(t) = 15\cos\left(\frac{\pi t}{12}\right) + 25$$

where h_B is the height in metres t hours after 9 am, on the same day as Station A.

c. How many metres was the **average height** of the river at Station A more than that at Station B over the interval $t \in [0,18]$? 2 marks

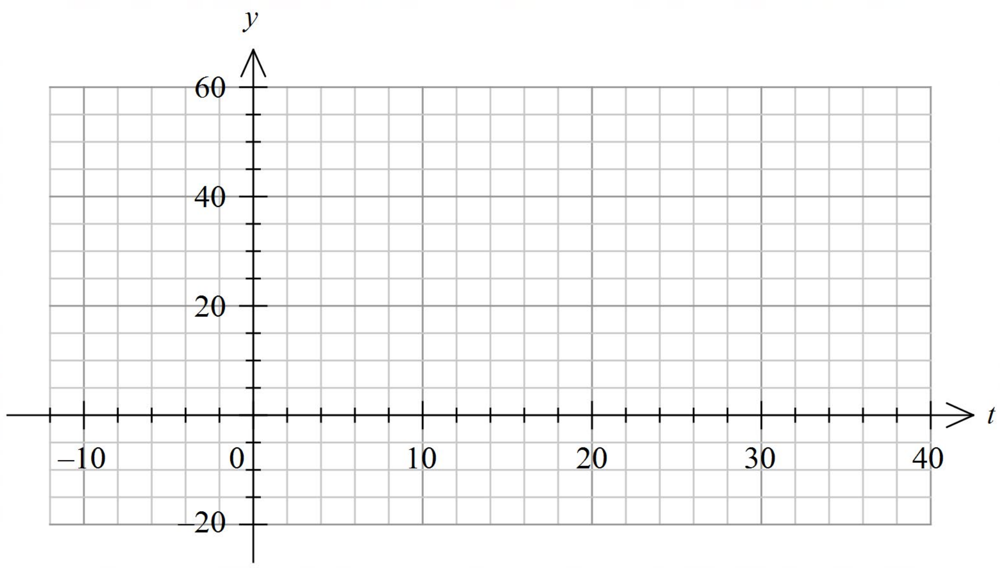
d. During the first 24 hours, what were the heights, in metres, of the river at Station A, when the height of the river was changing fastest at Station B? 2 marks

e. What is the period of the function, in hours, which represents the difference between the heights of the river at the two stations? Give the maximum difference, in metres correct to two decimal places. Assume the models continue for more than 24 hours. 2 marks

After 27 hours the tidal height at **Station A** changes, and from 9 am is modelled by the piecewise function w, defined by

$$w(t) = \begin{cases} h_A(t) & 0 \le t \le 27 \\ 30 & 27 < t \le 40 \end{cases}.$$

f. Sketch the graph of the piecewise function $y = w(t)$ on the axes below, showing the coordinates of any sharp points and endpoints. 2 marks



g. State the domain for which w is differentiable. 1 mark

After 40 hours the tidal height changes again and the height, p metres, at time t hours after 9 am on a Sunday of a certain week can be modelled by the function

$$p(t)=\begin{cases} w(t) & 0\le t\le 40 \\ m\cos\left(n(t-r)\right)+s & 40<t\le k \end{cases}$$ where $m,\ n,\ r,\ s$ and k are real constants.

Recording capacities for this river break down at 9 pm on Tuesday of that particular week and no records of height are taken after this time.

h. State the value of k . 1 mark

The graph of $p(t)$ is continuous and smooth at $t=40$ with the section described by $p=m\cos\left(n(t-r)\right)+s$ completing two cycles before recording capacities break.
Surprisingly for the recorders, the river over the domain $t\in\left[0,k\right]$ reaches zero height twice.

i. State a possible set of values for $m,\ n,$ and s and give r as a general solution for those values. 3 marks

Question 3 (16 marks)
There are three living species of elephants: the African bush elephant, the African forest elephant and the Asian elephant.

The masses of adult male African forest elephants are normally distributed. The probability that an adult male African forest elephant has a mass less than 5079 kg is 0.2 and a mass greater than 6141 kg is 0.1.

a. Find the mean and standard deviation of the masses of adult male African forest elephants. Give your answers in kilograms correct to one decimal place. 2 marks

The masses of adult male Asian elephants and adult male African bush elephants are also normally distributed. The means and standard deviations of the masses are shown in the table below.

	mean (kg)	standard deviation (kg)
Asian elephants	4085	445
African bush elephants	5375	225

An adult male elephant was randomly chosen from each of the three species and put in an enclosure.

b. If an elephant is randomly chosen from the enclosure and its mass is greater than 5079 kg, what is the probability, correct to four decimal places, it is the Asian elephant? 2 marks

SECTION B – Question 3 – continued

Rangers decided to check the health of the African bush elephants. They randomly select 20 of the elephants from sub-Saharan Africa, where they live, to make observations.

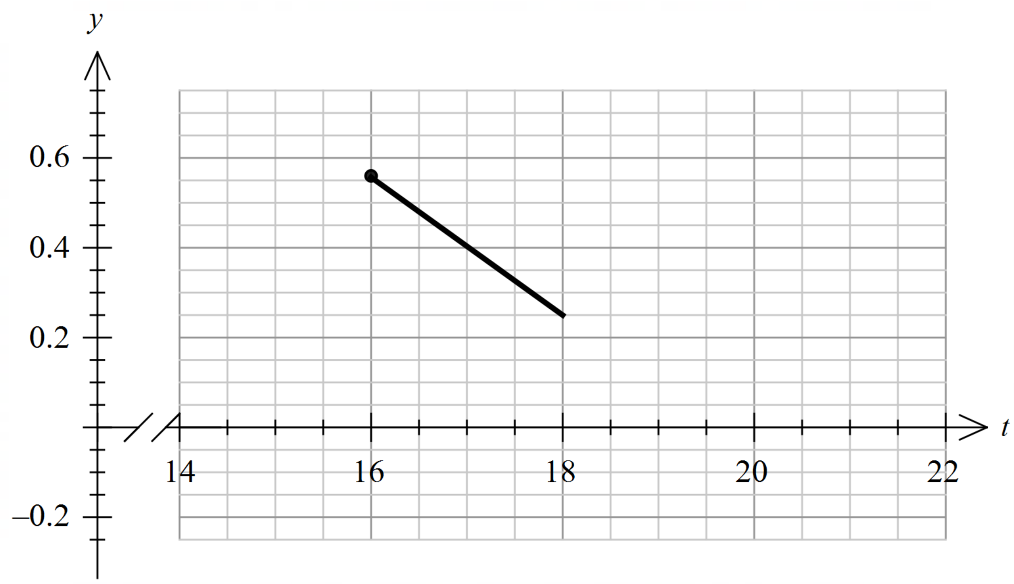
c. What is the probability, correct to four decimal places, that more than five of them weigh at least 5450 kg? 2 marks

d. What is the least number of African bush elephants the rangers would need to select to ensure the probability that more than five of them weigh at least 5450 kg is more than 0.95? 2 marks

Elephants spend more than two thirds of their day feeding on grasses, tree bark, roots and leaves. Let $f(t)$ be the probability density function for the number of hours, T, an elephant feeds per day.

$$f(t)=\begin{cases}\dfrac{\log_e(2)-1}{2}t+\dfrac{37-36\log_e(2)}{4} & 16\le t\le 18\\[2ex] \dfrac{1}{t-16}-\dfrac{1}{4} & 18<t\le 20\\[2ex] 0 & \text{elsewhere}\end{cases}$$

Part of the graph of $y=f(t)$ is shown below.



e. Sketch the remainder of the graph on the set of axes above. 1 mark

SECTION B – Question 3 – continued
TURN OVER

f. Find the standard deviation of T.
Give your answer correct to three decimal places. 2 marks

Elephants in captivity like to play with toys such as soccer balls and large beach balls. They enjoy batting the balls around with their trunks. In the wild they like chasing and playing with other animals such as zebras and giraffes.

A zoo keeper noticed that a particular brand of soccer ball was not lasting as long as the others. He contacted the company who donated the balls. The owner of the company randomly selected 30 balls and found 4 of them to be defective.

g. Based on this sample proportion, determine a 95% confidence interval for the proportion of balls that are likely to be defective.
Give your answers correct to four decimal places. 1 mark

In order to make a more reliable decision about whether to continue to donate the balls, the company owner wants to reduce the margin of error (half the confidence interval) to less than 10%.

h. Using a 95% level of confidence and $\hat{p} = \frac{2}{15}$, determine the size of the smallest sample the owner of the company must choose in order to achieve this goal. 1 mark

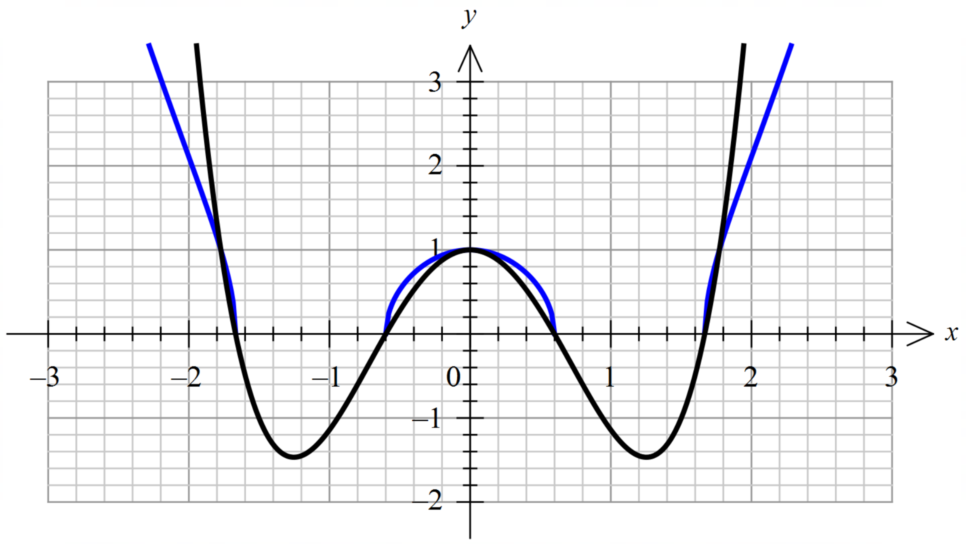
On a particular day, the probability that an African bush elephant and also an African forest elephant will chase a giraffe in the wild is $\frac{2-k^2}{2}$, where $0 < k < \sqrt{2}$. The two events are independent of each other. The probability that only the African forest elephant will make the chase is $\frac{k^2}{3}$.

i. Find the probability that neither elephant will make the chase.
Give your answer in terms of k. 2 marks

j. Find the maximum probability that only the African bush elephant will make the chase. 1 mark

Question 4 (14 marks)

The marketing team for a chain of supermarkets, Woolings, is designing a new big W logo. The front view of the logo is modelled by the functions *h* and *f* where $h(x)=\sqrt{x^4-px^2+1}$ and $f(x)=x^4-px^2+1$, and $p\in R$. *x* is the horizontal position in metres from *O* and *h* and *f*, the vertical position in metres from *O*. Part of the graphs of *h* and *f* are shown below for $p=3$.



a. Find the *x*-coordinates of the points of intersection of the graphs of *h* and *f* when $p=3$. 1 mark

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b. Find the *x*-coordinates of the points of intersection of the graphs of *h* and *f* in terms of *p*. 1 mark

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c. Using **part b.** explain why there are only 5 points of intersection when $p=2$. 1 mark

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d. Complete the table below for the number of points of intersection. 2 marks

Number of points of intersection	1	3
***p* values**		
***x*-coordinates of points of intersection**		

e. If $g(x)=\sqrt{x}$ and f exist over their maximal domains, for what values of p will $g\left(f(x)\right)$ **not** exist. Explain, giving appropriate values in terms of p. 1 mark

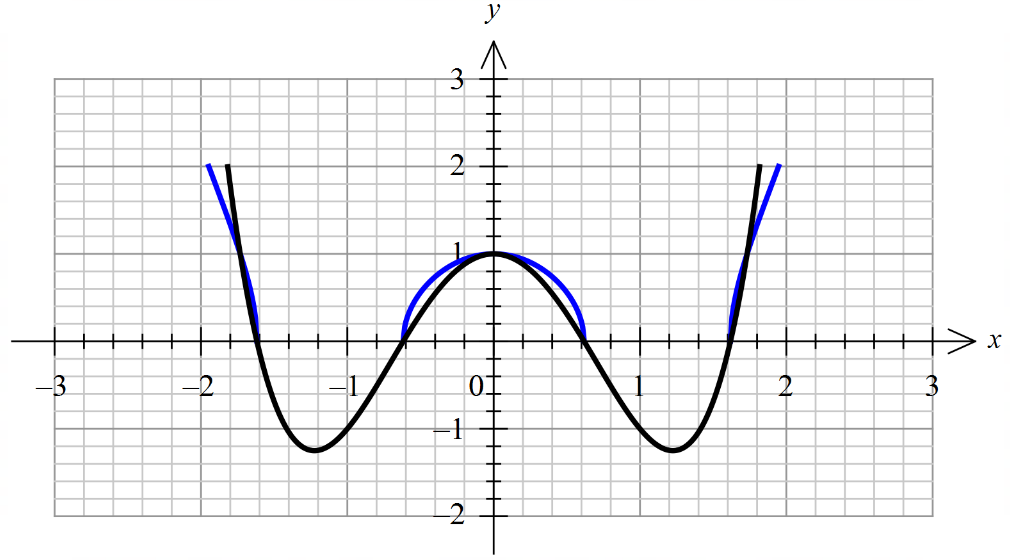
The marketing team want to use the design created when $p>2$ and to further investigate the properties of these functions.

f. For $p=3$, find the area of the region bound by f and h. Give your answer correct to three decimal places. 2 marks

g. Write an expression, in terms of p, using the sum of two definite integrals that will give the area of the region bound by f and h for any $p>2$. 1 mark

h. Find the value of p so that the points of inflection of f are on the x-axis. 2 marks

The company wishes to produce identical vases for their flower displays using the shape of the logo. Each vase will be a prism with a **cross-sectional** area modelled by the functions h_v and f_v where $h_v(x) = \sqrt{x^4 - 3x^2 + 1}$ and $f_v(x) = x^4 - 3x^2 + 1$ as shown below. x is the horizontal position in decimetres from O and h_v and f_v, the vertical position in decimetres from O. The maximum vertical position will be 2 dm from O and the length of each vase will be 3 dm. The cross-sectional area shows the height and width of each vase.



Each vase will be filled with water by a machine at a rate given by the function $\frac{dv}{dt} = 2^t$,

where v is the volume of water in the vase in dm^3 at time t seconds, $t \geq 0$.

i. How long will it take to fill each vase?
Give your answer in seconds correct to two decimal places. 3 marks

END OF QUESTION AND ANSWER BOOK