Question 1

a. $y = \dfrac{\log_e(x)}{x^2}$

$\dfrac{dy}{dx} = \dfrac{x^2 \times \dfrac{1}{x} - \log_e(x) \times 2x}{x^4}$

$\quad = \dfrac{x - 2x\log_e(x)}{x^4}$

$\quad = \dfrac{1 - 2\log_e(x)}{x^3}$ **1A**

b. $g(x) = x\tan^2(x)$

$g'(x) = x \times 2\tan(x)\sec^2(x) + \tan^2(x) \times 1$ **1M**

$g'\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{4} \times 2\tan\left(\dfrac{\pi}{4}\right)\sec^2\left(\dfrac{\pi}{4}\right) + \tan^2\left(\dfrac{\pi}{4}\right)$

$\quad = \dfrac{\dfrac{\pi}{2}}{\cos^2\left(\dfrac{\pi}{4}\right)} + 1$

$\quad = \dfrac{\dfrac{\pi}{2}}{\dfrac{1}{2}} + 1$

$\quad = \pi + 1$ **1A**

Question 2

$\log_e\left(3x^2 - 1\right) - \log_e\left(1 - 3x\right) = \log_e(2)$

$\log_e\left(\dfrac{3x^2 - 1}{1 - 3x}\right) = \log_e(2)$ **1M**

Equate $\dfrac{3x^2 - 1}{1 - 3x} = 2$

$3x^2 - 1 = 2 - 6x$

$3x^2 + 6x - 3 = 0$

$3\left(x^2 + 2x - 1\right) = 0$ **1M**

$x^2 + 2x - 1 = 0$

$x = \dfrac{-2 \pm \sqrt{4 + 4}}{2}$

$\quad = \dfrac{-2 \pm 2\sqrt{2}}{2}$

$$= -1 \pm \sqrt{2}$$

Reject $x = -1 + \sqrt{2}$ as $1 - 3x > 0$, $x < \dfrac{1}{3}$

For real solution $x = -1 - \sqrt{2}$ **1A**

Question 3

$$-2x + ky = m \Rightarrow y = \frac{2x + m}{k}$$

$$\left(1 + k^2\right)x + y = 2 \Rightarrow y = -\left(1 + k^2\right)x + 2$$

Equate gradients $\dfrac{2}{k} = -\left(1 + k^2\right)$ **1M**

$$1 + k^2 + \frac{2}{k} = 0$$

$$k^3 + k + 2 = 0$$

Using factor theorem

$$\left(k + 1\right)\left(k^2 - k + 2\right) = 0$$

$\Delta < 0$ for $k^2 - k + 2$ giving no real solution

$k = -1$ **1M**

Substitute $k = -1$ in equations

$$-2x - y = m$$

$$2x + y = 2$$

Values of k and m for which there are no solutions

$k = -1$, $m \neq -2$

Answer: $k = -1$, $m \in R \setminus \{-2\}$ **1A**

Question 4

$2\sin^2\left(x\right) + \sin(x) = 1$ for $x \in \left[-\pi, \pi\right]$

$$2\sin^2\left(x\right) + \sin(x) - 1 = 0$$

$$\left(2\sin\left(x\right) - 1\right)\left(\sin\left(x\right) + 1\right) = 0$$

$2\sin\left(x\right) = 1$, $\sin\left(x\right) = -1$ **1M**

$\sin\left(x\right) = \dfrac{1}{2}$ for $x \in \left[-\pi, \pi\right]$ gives $\qquad\qquad\sin\left(x\right) = -1$ for $x \in \left[-\pi, \pi\right]$ gives

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ **1A** $\qquad\qquad\qquad\qquad x = -\dfrac{\pi}{2}$ **1A**

Question 5

$f : R \setminus \{2\} \to R, f(x) = \dfrac{1}{\left(2 - x\right)^2}$ and $g : \left[-2, k\right) \to R, g(x) = 2x + 1$

a. For $k = 1$, $g : \left[-2, 1\right) \to R, g(x) = 2x + 1$

For $f \circ g$ to exist, test range $(g) \subseteq$ dom (f)

$\left[-3, 3\right) \not\subseteq R \setminus \{2\}$

$f \circ g$ does not exist **1A**

b. For $f \circ g$ to exist

$[-3, 2k+1) \in R \setminus \{2\}$

$2k+1 = 2 \Rightarrow k = \dfrac{1}{2}$ such that $f \circ g$ exists   **1A**

c. From **part b.** $k = \dfrac{1}{2}$

Dom $f \circ g : \left[-2, \dfrac{1}{2}\right)$

Rule $(f \circ g)(x) = \dfrac{1}{(1-2x)^2}$ **1A** both rule and domain

Question 6

a. $\displaystyle\int_2^3 \dfrac{3}{2x-1}\,dx = \dfrac{3}{2}\int_2^3 \dfrac{2}{2x-1}\,dx$

$= \dfrac{3}{2}\left[\log_e(2x-1)\right]_2^3$ **1M**

$= \dfrac{3}{2}\left(\log_e(5) - \log_e(3)\right)$

in the form $a\log_e(b)$ where $a, b \in Q$

$\displaystyle\int_2^3 \dfrac{3}{2x-1}\,dx = \dfrac{3}{2}\log_e\left(\dfrac{5}{3}\right)$ **1A**

b. Two trapeziums of width $\dfrac{1}{2}$

Approximate area $= \dfrac{\frac{1}{2}}{2}\left(f(2) + 2f\left(\dfrac{5}{2}\right) + f(3)\right)$

$= \dfrac{1}{4}\left(\dfrac{3}{2(2)-1} + 2\left(\dfrac{3}{2\left(\frac{5}{2}\right)-1}\right) + \dfrac{3}{2(3)-1}\right)$ **1M**

$= \dfrac{1}{4}\left(1 + \dfrac{3}{2} + \dfrac{3}{5}\right)$

$= \dfrac{1}{4} \times \dfrac{31}{10}$

Approximate area $= \dfrac{31}{40}$ **1A**

c. $\dfrac{31}{40} > \dfrac{3}{2}\log_e\left(\dfrac{5}{3}\right)$

Area in **part b.** is an over estimate of the actual area because the upper straight side of each trapezia sits above the concave upward curve.   **1A**

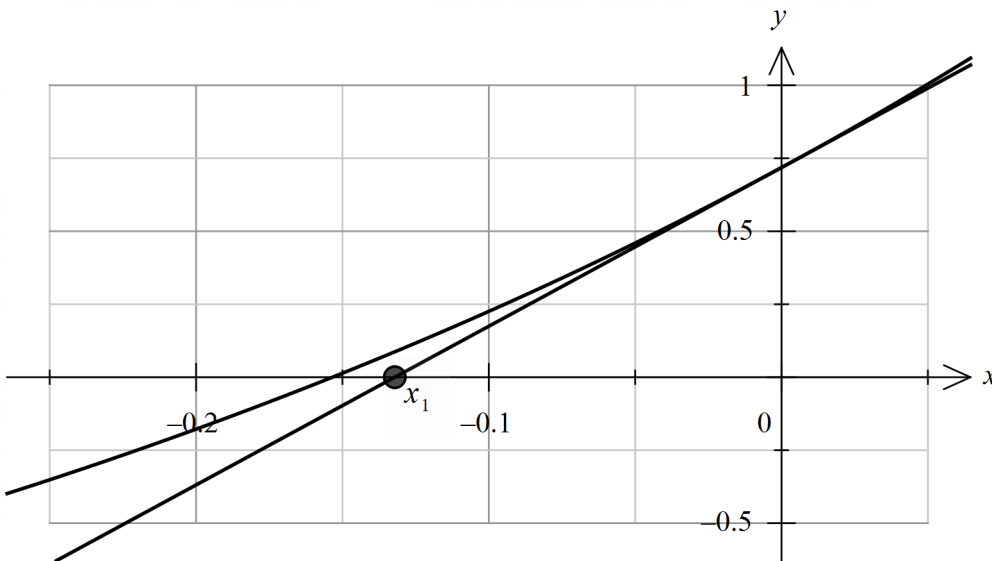**Question 7**

$f(x) = e^{2x+1} - 2$

a. $x_1 = x_0 - \dfrac{f(0)}{f'(0)}$

$x_1 = 0 - \dfrac{e^{2\times 0+1} - 2}{2e^{2\times 0+1}}$

$x_1 = -\dfrac{e-2}{2e} = \dfrac{2-e}{2e} = \dfrac{1}{e} - \dfrac{1}{2}$ **1A** or equivalent form

b. Tangent line with x_1 labelled.   **1A**

c. $e^{2x+1} - 2 = 0$

$e^{2x+1} = 2$

$2x + 1 = \log_e(2)$

$x = \dfrac{\log_e(2) - 1}{2}$ **1A**

Distance $= \dfrac{2-e}{2e} - \dfrac{\log_e(2) - 1}{2}$

$= \dfrac{2 - e\log_e(2)}{2e}$

$= \dfrac{1}{e} - \dfrac{\log_e(2)}{2}$

$= \dfrac{1}{e} + \log_e\left(\dfrac{1}{\sqrt{2}}\right)$ **1H**

Question 8

$f(x) = 1 + \sqrt{2x - 3}$

a. $f'(x) = \dfrac{1}{\sqrt{2x-3}} = 1$

$$\frac{1}{\sqrt{2x-3}}=1$$

$$1=\sqrt{2x-3}$$
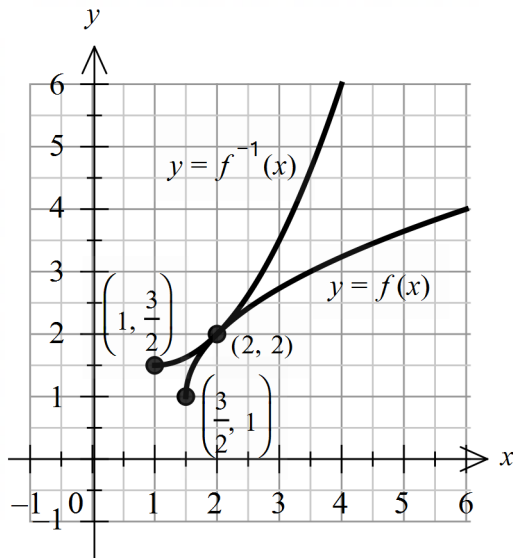
$$2x-3=1$$

$$x=2$$

$(2,2)$ coordinates   **1A**

b. Shape $y=f(x)$ **1A**

Shape $y=f^{-1}(x)$ **1A**

Coordinates   **1A**

c. $g(x)=1+\sqrt{2x-a}=1+\sqrt{2\left(x-\frac{a}{2}\right)}$

As the graph of f touches the line $y=x$ then $a<3$.

The graphs of g will intersect the line $y=x$ twice until the coordinates of the endpoint are $(1,1)$.

If the graph of f is translated $\frac{1}{2}$ a unit left it will intersect the line $y=x$ at $(1,1)$.

$$g(x)=1+\sqrt{2\left(x-\frac{3}{2}+\frac{1}{2}\right)}=1+\sqrt{2(x-1)}=1+\sqrt{2x-2}$$

Hence $2\le a<3$.   **1A**

OR

Solve $1+\sqrt{2x-a}=x$

$$\sqrt{2x-a}=x-1$$

$$2x-a=(x-1)^2$$

$$2x-a=x^2-2x+1$$

$$x^2-4x+1+a=0$$

$$\Delta=16-4(1+a)>0$$

$$12-4a>0$$

$$a<3$$

Substitute $x=1$ into $x^2-4x+1+a=0$ to find the minimum value of a.

$$a=2$$

Hence $2 \le a < 3$.   **1A**
OR
Substitue $(1,1)$ into $g(x) = 1 + \sqrt{2x - a}$
$1 + \sqrt{2 - a} = 1$
$\sqrt{2 - a} = 0$
$a = 2$
Hence $2 \le a < 3$.   **1A**

d. If the graph of f is translated $\dfrac{1}{2}$ a unit left then the points of intersection are at $x = 1$ and $x = 3$.

OR
Solve $1 + \sqrt{2x - 2} = x$
$2x - 2 = (x - 1)^2$
$x^2 - 4x + 3 = 0$
$(x - 1)(x - 3) = 0$
$x = 1$ or $x = 3$

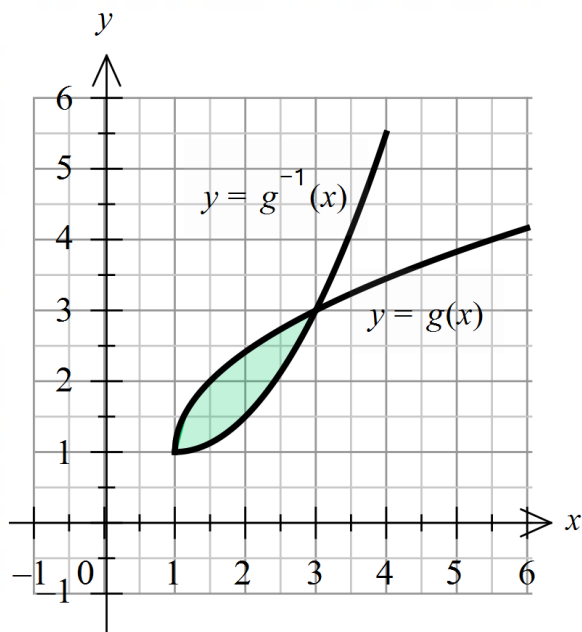$2 \displaystyle\int_1^3 \left(1 + \sqrt{2x - 2} - x\right) dx$ **1M**

$= 2 \left[x + \dfrac{1}{3}(2x - 2)^{\frac{3}{2}} - \dfrac{1}{2}x^2 \right]_1^3$ **1M**

$= 2 \left(\left(3 + \dfrac{8}{3} - \dfrac{9}{2}\right) - \left(1 - \dfrac{1}{2}\right) \right)$

$= 2 \left(-2 + \dfrac{8}{3} \right)$

$= \dfrac{4}{3}$ **1A** or equivalent form

Question 9

a. 0.02^3

$= 0.000\ 008$ or $\dfrac{1}{125\ 000}$ **1A**

b. $X \sim \mathrm{Bi}(4, 0.3)$ **1M**

$\Pr(X = 4) = \dbinom{4}{3} 0.3^3 \times 0.7$

$= 4 \times 0.027 \times 0.7$

$= 0.0756$ or $\dfrac{189}{2500}$ **1A**

c. $2 \times 0.02 \times 0.98$

$= 0.0392$ or $\dfrac{49}{1250}$ **1A**

d. $0.02 n_t = 0.3 n_s$

$n_t = 15 n_s$, $n_s \in Z^+$ **1A**

e. $d(t) = \begin{cases} t - 1 & 1 \le t \le 2 \\ a(t-3)^2 + b & 2 < t \le 3 \\ 0 & \text{elsewhere} \end{cases}$

$a(t-3)^2 + b = t - 1$ when $t = 2$

$a + b = 1 \ldots (1)$ **1A**

$\displaystyle\int_1^2 (t-1)\, dt + \int_2^3 \left(a(t-3)^2 + b \right) dt = 1$

$\left[\dfrac{t^2}{2} - t \right]_1^2 + \left[\dfrac{a}{3}(t-3)^3 + bt \right]_2^3 = 1$

$\dfrac{1}{2} + \left[\dfrac{a}{3}(t-3)^3 + bt \right]_2^3 = 1$

$\left[\dfrac{a}{3}(t-3)^3 + bt \right]_2^3 = \dfrac{1}{2}$

$b + \dfrac{a}{3} = \dfrac{1}{2} \ldots (2)$ **1M** attempt to set up the second equation

$a + b = 1 \ldots (1)$

Subtract (2) from (1)

$\dfrac{2}{3} a = \dfrac{1}{2}$

$a = \dfrac{3}{4}$, $b = \dfrac{1}{4}$ **1A**

END OF SOLUTIONS