VCE® Mathematical Methods

Unit 3 and 4 Practice Written Examination 2

SOLUTIONS

Solution Pathway

Below are sample answers. Please consider the merit of alternative responses.

Section A: Multiple-Choice Questions (20 marks)

Question	Correct Answer	Explanation
1	**B**	$Period = \frac{2\pi}{n} = \frac{2\pi}{3}$ $Amplitude = 2$
2	**D**	The function is smooth between [-2, 4]. The coordinates of the endpoints are (-2, 0) and (4, 16). However, the minimum value of $g(x)$ within this domain is -9. Therefore, the range is [-9, 16].
3	**E**	Normal line function on CAS. Alternatively. $\frac{dy}{dx} = (x - 2)(3x - 2)$ $At\ x = 0. \frac{dy}{dx} = 4$ $\therefore m_N = -\frac{1}{m} = -\frac{1}{4}$ $At\ x = 0,\ y = 0$ $y = mx + c$ $= -\frac{1}{4}x,\ since\ c = 0$
4	**E**	By CAS Define $f(x) = x^2 + bx + c$ Solve the following system of equations for b and c: $\{f'(1) = 4 \qquad \frac{1}{2}\int_1^3 f(x) = \frac{34}{3}$ $b = 2,\ c = 3$ $\therefore f(x) = x^2 + 2x + 3$ $f(2) = 11$

		Alternatively $f'(x) = 2x + b$ $f'(1) = 2 + b = 4$ $\therefore b = 2$ $f(x) = x^2 + 2x + c$ $y_{av} = \frac{1}{b-a}\int_a^b f(x)dx$ $\frac{34}{3} = \frac{1}{2}\int_1^3 x^2 + 2x + c\,dx$ $\frac{68}{3} = \left[\frac{1}{3}x^3 + x^2 + cx\right]_1^3$ $\frac{68}{3} = (9 + 9 + 3c) - \left(\frac{1}{3} + 1 + c\right)$ $\frac{68}{3} = 2c + \frac{50}{3}$ $2c = \frac{18}{3}$ $c = 3$ $f(x) = x^2 + 2x + 3$ $\therefore f(2) = 11$
5	**C**	Solve $\det det\,(A) = 0,\ A = [m\ 3\ 2\ m - 1\ ]$ $m =- 2\ or\ 3$ At these values of m the gradients of the lines are parallel. There will be no solutions if the y -intercepts of the lines are not equal. Evaluate the y -intercept of each line for the given m values. They are not equal in both cases. Therefore, there will be no solution when $m =- 2$ or $m = 3$.
6	**A**	$\Pr Pr\,(B) = \frac{Pr(A \cap B)}{Pr(B)}$ $\therefore \Pr Pr\,(X \geq 1) = \frac{Pr(X \geq 2 \cap X \geq 1)}{Pr(X \geq 1)}$

		$= \frac{Pr(X\geq 2)}{Pr(X\geq 1)}$ $(X\geq 2)\ \ and\ Pr(X\geq 1)$ $\Pr Pr\ (X\geq 2\ \vert p\ = \frac{5}{12},\ n\ =\ 3,\ lower\ =\ 2,\ upper\ 3)\ =\ 0.3762$ $\Pr Pr\ (X\geq 1\ \vert p\ = \frac{5}{12},\ n\ =\ 3,\ lower\ =\ 1,\ upper\ 3)\ =\ 0.8015$ $\therefore \Pr Pr\ (X\geq 1) = \frac{0.3762}{0.8016}$ $=\ 0.4693$

7	**C**	$\Pr Pr\,(X \geq 1)\ =\ 1\ -\ Pr(X\ =\ 0)$ $1\ -\ \Pr Pr\,(X\ =\ 0) \geq 0.9$ $1 - {}^{n}C_{r}p^{r}(1\ -\ p)^{n-r} \geq 0.9$ $r\ =\ 0,\ p\ =\ 0.3$ $1 - {}^{n}C_{0}0.3^{0}0.7^{n} \geq 0.9$ $1\ -\ 0.7^{n} \geq 0.9$ Solve for n $n \geq 6.46$ $\therefore n\ =\ 7$
8	**A**	Since this is a pdf, area under pdf = 1. $Solve\ for\ a\ on\ CAS\ \int_{2}^{a} \frac{1}{\sqrt{x}} dx\ =\ 1$ $a\ = \frac{4\sqrt{2}+9}{4}$ Alternatively: $\left[2\sqrt{x}\right]_{2}^{a} =\ 1$ $\left(2\sqrt{a}\right) - \left(2\sqrt{2}\right) =\ 1$ $2\sqrt{a} =\ 2\sqrt{2}\ +\ 1$ $\sqrt{a}\ = \frac{2\sqrt{2}+1}{2}$ $a\ = \left(\frac{2\sqrt{2}+1}{2}\right)^{2} = \frac{8+4\sqrt{2}+1}{4} = \frac{4\sqrt{2}+9}{4}$
9	**D**	Using symmetry of Normal distribution,$(X\ <\ 55)\ \ =\ 0.25$ Use standard Normal distribution to find the *Z*-score. $Z\ =-\ 0.6745$ $Z\ = \frac{x-\mu}{\sigma}$ $\therefore \sigma\ = \frac{x-\mu}{Z}$

		$= \frac{55-65}{-0.6745}$ $= 14.83$
10	**A**	Use Normal Cdf function on CAS to determine probability that a person sets up a tent in under 7 minutes. $\Pr Pr\ (X < 7 \vert \mu = 8.5, \sigma = 2.0) = 0.22\ 66$ A person will set up a tent in under 7 minutes, or they won't. Therefore, use Binomial Cdf of Pdf to calculate required probability. $\Pr Pr\ (p = 0.2266,\ n = 3)\ = 0.0116$
11	**D**	The intercepts of $f(x)$ indicate that $F(x)$ has turning points at negative *x*-coordinates. This eliminates options A, B and E, which each have a positive turning point at $x = 3$. Option E is a negative cubic, which means its gradient is negative beyond the turning point with the larger *x*-coordinate. However, the graph of $f(x)$ indicates that the gradient is positive in this region. Only option D remains, and it is consistent with $f(x)$ in terms of approximate position of turning points and gradient (positive/negative).
12	**C**	Since there are two *x*-intercepts and a turning point located above the *x*-axis, there must be a second turning point located on the x-axis. If this were not the case, there would be either 1 or 3 *x*-intercepts. Therefore, $f(x)$ has the following form when factorised: $f(x) = ax(x - h)^2$ Solve the following system of equations for a and h. $\{f'\left(\frac{4}{3}\right) = 0\ f\left(\frac{4}{3}\right) = 256$ $a = 27,\ b = 4$ Substitute these values into $f(x)$ above and expand. $\therefore a = 27,\ b =- 216,\ c = 432$
13	**B**	Solve the following system of equations for a and x. $\{f(x) = g(x)\ f'(x) = g'(x)$

		$a = 1.44$ to 2 decimal places. Note that the CAS may not solve the system of equations directly. One approach is to use a substitution method. For example, solve $f'(x) = g'(x)$ for a, then substitute the expression for a into $f(x) = g(x)$.

14	**E**	Test each option. A is incorrect since the middle term will be negative. B is incorrect since the middle term will be negative. C is incorrect since the last term will be negative. D is incorrect since the first term includes regions above and below the *x*-axis. E is correct since each term correctly takes into account the sign of the area and the terminals match the boundaries.
15	**C**	$f(x) = 2\cos cos\,(4x) + 1$ Reflection about the *x*-axis. $-f(x) = -2\cos cos\,(4x) - 1$ Dilation by a factor of 2 from *y*-axis. $-f\left(\frac{1}{2}x\right) = -2\cos cos\,(2x) - 1$ Translation $\frac{\pi}{2}$ to the right. $-f\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right) = -2cos\left(2\left(x - \frac{\pi}{2}\right)\right) - 1$
16	**B**	Define each function on the CAS then evaluate each of the equations. Only option B is true.
17	**A**	$l_1 = 1 - x$ $Let\ L\ be\ the\ length\ of\ l_2,\ the\ distance\ from\ l_1 to\ the\ origin.$ $L = \sqrt{\left(y_2 - y_1\right)^2 + \left(x_2 - x_1\right)^2}$ $= \sqrt{(1 - x - 0)^2 + (x - 0)^2}$ $= \sqrt{2x^2 - 2x + 1}$ $L_{av} = \frac{1}{b-a}\int_a^b L(x)dx$ $= \int_0^1 \sqrt{2x^2 - 2x + 1}dx$

		$= \frac{\sqrt{2}(2\sqrt{2}+3)}{8} + \frac{1}{2}\ units$

18	**E**	$Width = Upper\ Bound - Lower\ Bound$ $= 2Z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ $\hat{p} = \frac{8}{11}$ Use Inverse Normal Cdf to find Z (area = 0.975) for a 95% confidence interval. $Z = 1.96$ $0.114 = 2\times1.96\times\sqrt{\frac{\frac{8}{11}\times\frac{3}{11}}{n}}$ Solve for n. $n = 235$
19	**D**	$By\ symmetry\ of\ the\ C.I,\ \hat{p} = \frac{0.642+0.708}{2} = 0.675$ $Width = Upper\ Bound - Lower\ Bound$ $= 0.708 - 0.642$ $= 0.066$ $0.066 = 2\times Z\times\sqrt{\frac{0.675(1-0.675)}{542}}$ Solve for Z $Z = 1.64$

20	**C**	Consider $f \circ g$ A. $f(x) = \sqrt{x},\ g(x) = tan(x)$ Check $Ran(g(x)) \subseteq Dom(f(x))$ $R \subseteq R^{+} \cup \{0\}$ False B. $f(x) = (x),\ \ g(x) = x^{2}$ Check $Ran(g(x)) \subseteq Dom(f(x))$ $[0, \infty) \subseteq R^{+}$ False C. $f(x) = e^{x}, g(x) = cos(x)$ Check $Ran(g(x)) \subseteq Dom(f(x))$ $[-\ 1, 1] \subseteq R$ True D. $f(x) = x^{-2}, g(x) = (x)$ Check $Ran(g(x)) \subseteq Dom(f(x))$ $R \subseteq R \backslash \{0\}$ False E. $f(x) = \sin sin\,(x)\ ,\ g(x) = \sqrt{x}$ Check $Ran(g(x)) \subseteq Dom(f(x))$ $R^{+} \cup \{0\} \subseteq R$ True Therefore options A,B,D can be ignored Consider $g \circ f$ C. $f(x) = e^{x}, g(x) = cos(x)$ Check $Ran(f(x)) \subseteq Dom(g(x))$ $R^{+} \subseteq R$ True E. $f(x) = \sin sin\,(x)\ ,\ g(x) = \sqrt{x}$ Check $Ran(f(x)) \subseteq Dom(g(x))$ $[-\ 1, 1] \subseteq R^{+} \cup \{0\}$ False Only option C is defined for both $f \circ g$ and $g \circ f$.

SECTION B (60 marks)

Question 1 (12 marks)

A family of curves is defined by the equation:

$$f(x) = a\sqrt{x} + \frac{b}{x}, \ \ a, b \in R^+$$

a. What is the implied domain of $f(x)$? 1 mark

$Dom(f(x)) = Dom(a\sqrt{x}) \cap Dom\left(\frac{b}{x}\right)$

$= (R^+ \cup 0) \cap (R \backslash \{0\})$

$= R^+$

1 mark for correct implied domain expressed in valid form.

b. Show that $f(x)$ has no x-intercepts. 2 marks

$f(x) = 0$

$a\sqrt{x} + \frac{b}{x} = 0$

$a\sqrt{x} = -\frac{b}{x}$

But LHS is positive, since $a > 0$ and $\sqrt{x} > 0$, and RHS is negative, since $b > 0$ and $x > 0$ and therefore $-\frac{b}{x} < 0$. Since LHS ≠ RHS for any $x > 0$, no solutions and hence no x-intercepts.

1 mark for a valid mathematical approach.
1 mark for a valid conclusion based upon approach used.

c. Consider the case where a = 1 and b = 1.

(i) Determine the x-coordinate of the turning point. 1 mark

$f(x) = \sqrt{x} + \frac{b}{x}$

Solve $f'(x) = 0$

$x = 4^{\frac{1}{3}} = \sqrt[3]{4} = 2^{\frac{2}{3}}$

1 mark for correct x-value of the turning point.

(ii) Show that the turning point is local minimum. **1 mark**

Check gradient of the function to the left and right of the turning point.

$f'(1) = -\frac{1}{2} < 0$

$f'(2) = \frac{\sqrt{2}-1}{4} > 0$

Since the gradient is negative to the left of the turning point and positive to the right, the turning point is a local minimum.

1 mark for a valid approach for showing that the turning point is a local minimum.

(iii) Determine the exact value of the *y*-coordinate of the turning point. **1 mark**

$f\left(4^{\frac{1}{3}}\right) = 2^{\frac{1}{3}} + \frac{1}{4^{\frac{1}{3}}} = \sqrt[3]{2} + \frac{1}{\sqrt[3]{4}}$

1 mark for the correct *y*-coordinate of the turning point.

(iv) What is the equation of the vertical asymptote? **1 mark**

$x = 0$

1 mark for the correct equation of the vertical asymptote.

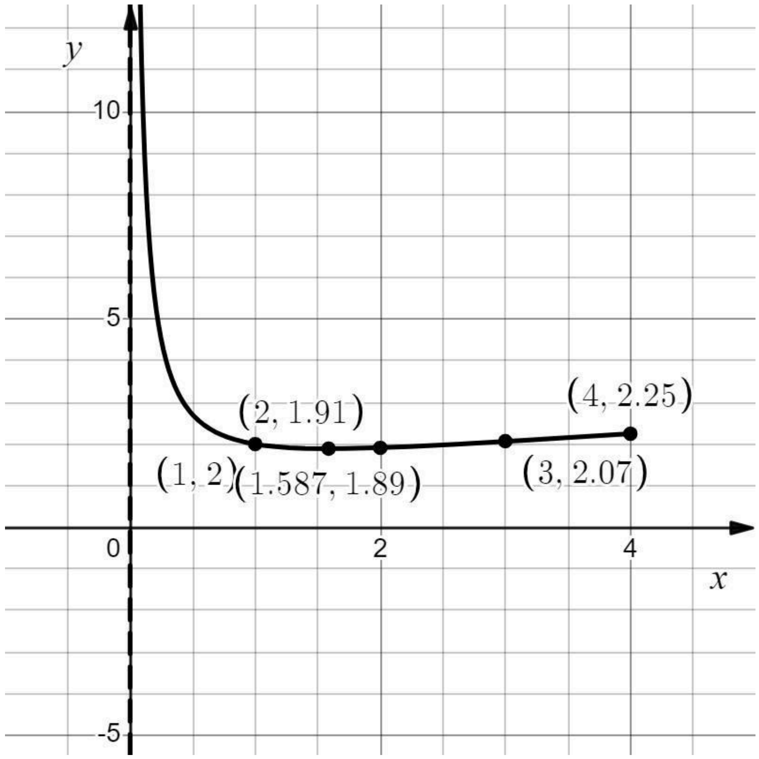
(v) Use addition of ordinates and the information developed in the preceding parts to sketch the graph of:

$$g: R \rightarrow (0, 4], g(x) = \sqrt{x} + \frac{1}{x}$$

Include the equations of any horizontal and vertical asymptotes, as well as the coordinates of any axes intercepts and turning point. **3 marks**

Reference points using addition of ordinates.

$g(4) = \frac{9}{4}$, $g(1) = 2$, $g(2) = \sqrt{2} + \frac{1}{2} \sim 1.91$, $g(3) = \sqrt{3} + \frac{1}{3} \sim 2.07$



1 mark for reference points (evidence of addition of ordinates).
1 mark for asymptote and turning point labelled.
1 mark for correct shape including end point at $x = 4$ correctly indicated.

d. Under what conditions will the turning point of $f(x)$ always occur at $x = 2$? 2 marks

$f'(x) = \frac{a}{2\sqrt{x}} - \frac{b}{x^2}$

$f'(2) = \frac{a}{2\sqrt{2}} - \frac{b}{4}$

$\frac{a}{2\sqrt{2}} - \frac{b}{4} = 0$

$\frac{a}{2\sqrt{2}} = \frac{b}{4}$

$a = \frac{b\sqrt{2}}{2}\ or\ b = \sqrt{2}a$

The turning point will always occur at $x = 2$ when $a = \frac{b\sqrt{2}}{2}\ or\ b = \sqrt{2}a$.

1 mark for a valid application of the constraint that $x = 2$.
1 mark for a valid condition.

Question 2 (12 marks)

Consider the function $f(x) = e^{bx}, b \in R^{+}$

a. Determine the rule for $f^{-1}(x)$, the inverse of $f(x)$. State the domain and range of $f^{-1}(x)$. **2 marks**

$Solve\ x = e^{by}\ for\ y$

$y = \frac{1}{b}(x)$

$\therefore f^{-1}(x) = \frac{1}{b}(x)$

$Dom\left(f^{-1}(x)\right) = Ran(f(x)) = R^{+}$

$Ran\left(f^{-1}(x)\right) = Dom(f(x)) = R$

1 mark for correct inverse.
1 mark for correct domain and range of the inverse expressed in a valid form.

b. What is the gradient of $f(x)$ for when $f(x)$ and $f^{-1}(x)$ intersect once? **1 mark**

$f(x)$ and $f^{-1}(x)$ intersect on the line of $y = x$. If there is one point of intersection then $f(x)$ is tangential to $y = x$, which means its gradient is 1.

1 mark for correct inverse.

c. Hence determine the point of intersection of $f(x)$ and $f^{-1}(x)$ for when $f(x)$ and $f^{-1}(x)$ intersect once only. **2 marks**

$Solve\ f^{'}(x) = 1$

$be^{bx} = 1$

$x = \left(\frac{1}{b}\right)$

Substituting this value into $f(x)$ gives:

$f\left(\left(\frac{1}{b}\right)\right) = \frac{1}{b}$

However, since the point of interest lies along the line $y = x$ it follows that:

$\left(\frac{1}{b}\right) = \frac{1}{b}$

Solving for b gives:

$b = \frac{1}{e}$

$f(x)$ and $f^{-1}(x)$ will intersect once when $b = \frac{1}{e}$.

$Since\ f\left(\frac{1}{e}\right) = \frac{1}{e},\ f^{-1}\left(\frac{1}{e}\right) = \frac{1}{e}$

$When\ there\ is\ one\ point\ of\ intersection\ the\ point\ is \left(\frac{1}{e}, \frac{1}{e}\right)$.

1 mark for a valid expression of x in terms of b.
1 mark for correct point of intersection.

d. Determine the values of b for when $f(x)$ and $f^{-1}(x)$ do not intersect and intersect at 2 points. **2 marks**

Testing larger values of b shows that there is no point of intersection. Testing smaller positive values of b shows that there are two points of intersection.

$0\ solutions\ for\ b > \frac{1}{e}$

$2\ solutions\ for\ 0 < b < \frac{1}{e}$

1 mark for correct value of b for when there is no point of intersection.
1 mark for correct value of b for when there are two points of intersection.

e. Determine the minimum distance between $f(x)$ and $f^{-1}(x)$, in terms of b, under the condition that $f(x)$ and $f^{-1}(x)$ do not intersect. Hint: use properties of inverses. **3 marks**

The minimum distance is the length of a line segment connecting $f(x)$ and $f^{-1}(x)$ and which is perpendicular to both. Also, since $f(x)$ and $f^{-1}(x)$ are reflected about the line $y = x$, the line segment will be perpendicular to this line. Thus, the gradient of the line segment is -1. This means the gradient of $f(x)$ is +1.

$Solve\ f'(x) = 1\ for\ x$

$x = -\frac{(b)}{b}$

$\therefore f\left(-\frac{(b)}{b}\right) = \frac{1}{b}$

$The\ shortest\ line\ segment\ will\ intersect\ f(x)\ at\ \left(-\frac{(b)}{b}, \frac{1}{b}\right)$

$Since\ the\ gradient\ of\ the\ line\ segment\ is\ -1,\ it\ will\ intersect\ f^{-1}(x)\ at\ \left(\frac{1}{b}, -\frac{(b)}{b}\right)$

Let *L* be the length of this line.

$$L = \sqrt{\left(y_2 - y_1\right)^2 + \left(x_2 - x_1\right)^2}$$

$$= \sqrt{\left(-\frac{(b)}{b} - \frac{1}{b}\right)^2 + \left(\frac{1}{b} + \frac{(b)}{b}\right)^2}$$

$$= \sqrt{2\left(\frac{1}{b} + \frac{(b)}{b}\right)^2}$$

$$= \sqrt{2}\left(\frac{1+(b)}{b}\right)$$

1 mark for identifying point of intersection of shortest line segment on $f(x)$.
1 mark for identifying point of intersection of shortest line segment on $f^{-1}(x)$.
1 mark for a correct equation for the shortest distance in terms of b.

f. **Hence determine the value of b for when this length is a maximum and state the value of this length.** **2 marks**

$Solve\ L'(b) = 0\ for\ b$

$b = 1$

Maximum length of the shortest distance occurs when $b = 1$.

The value of this length is $L(1)$.

$L(1) = \sqrt{2}$ units.

1 mark for correct value of b.
1 mark for correct length.

Question 3 (15 marks)

a. For a given angle, θ, what is the initial height of the swing seat above the ground?
1 mark

$initial\ height = l - l\cos cos\,(\theta)$

1 mark for a correct equation for the height of the swing seat above the ground.

b. The height, h (meters), of the seat above the ground, as a function of time, can be modelled by the equation:

$$h(t) = a\cos cos\,(nt) + k, \quad a, n, k \in R$$

a, n and k are model parameters that depend upon l and θ.

(i) Determine the value of a in the model. **1 mark**

The height will oscillate between the initial, maximum, height and zero. The amplitude, *a*, of this function will therefore correspond to half of this maximum.

$a = \frac{l - l\cos cos\,(\theta)}{2}$

1 mark for a correct expression for the value of *a*.

(ii) Determine the value of n in the model. **1 mark**

$The\ period\ of\ the\ function\ is\ given\ by\ P = 2\pi\sqrt{\frac{l}{10}}$

$P = \frac{2\pi}{n}\ for\ the\ cosine\ function.$

$n = \frac{2\pi}{P}$

$= \frac{2\pi}{2\pi\sqrt{\frac{l}{10}}}$

$= \sqrt{\frac{10}{l}}$

1 mark for a correct expression for the value of n.

(iii) Determine the value of k in the model. **1 mark**

The minimum value of l, corresponding to local minima, occurs at height = 0.

$h(t)$ therefore needs to be translated vertically by an amount equal to the amplitude, a.

$$k = \frac{l - l\cos cos\,(\theta)}{2}$$

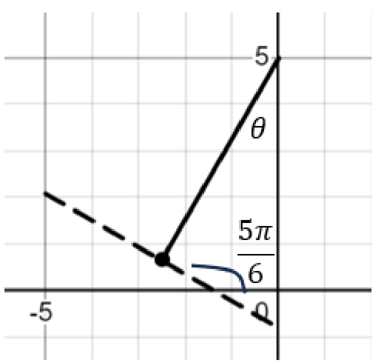
1 mark for a correct expression for k.

(iv) Hence, express the model in terms of l and θ and t. **1 mark**

$$h(t) = \frac{l - l\cos cos\,(\theta)}{2} \cos cos \left(\sqrt{\frac{10}{l}}t\right) + \frac{l - l\cos cos\,(\theta)}{2}$$

1 mark for correct equation.

(v) For a particular swing, l is 5 meters. If the angle of the tangent to the arc described by the swing seat at *t* = 0 is $\frac{5\pi}{6}$ radians. Determine the initial height of the swing above the ground. **2 marks**



The diagram above describes the situation. The angle the tangent makes with the y-axis (the vertical line where the swing is attached) is $\frac{\pi}{3}$. Therefore the angle the swing makes with the vertical is $\frac{\pi}{6}$.

$$h(t) = \frac{l - l\cos cos\,(\theta)}{2} \cos cos \left(\sqrt{\frac{10}{l}}t\right) + \frac{l - l\cos cos\,(\theta)}{2} \Big| t = 0,\ l = 5,\ \theta = \frac{\pi}{6}$$

$$= 5 - \frac{5\sqrt{3}}{2} m$$

Alternatively:

$h = l - l \cos cos\,(\theta)|l = 5,\ \theta = \frac{\pi}{6}$

$= 5 - \frac{5\sqrt{3}}{2}m$

1 mark for correct value of θ.
1 mark for correct height.

c. Let a be the dilation factor of the end sections and b be the dilation factor of the middle section.

(i) What is the equation for the section between points O and B? 1 mark

$y = ax^2$

There is no vertical or horizontal translation.

1 mark for the correct equation.

(ii) What is the equation for the section between points B and D? 1 mark

Horizontal translation of 15 and a vertical translation of 10. The dilation factor is b.

$y = b(x - 15)^2 + 10 \quad or\ y =- b(x - 15)^2 + 10$

Either equation is acceptable since the either will allow for the correct equation when parameter values are determined. In the first case b will be negative whereas in the second case it will be positive but will appear negative in the equation due to the negative sign.

1 mark for a valid equation.

(iii) Express a in terms of b such that two of the sections joins smoothly at point B. Express the relationship in the form $a = f(b)$. **3 marks**

To join smoothly, the first two sections must join at the same point and have the same gradient.

Let $f(x)$ and $g(x)$ represent the first and middle sections respectively.

$f(x) = ax^2$

$g(x) = b(x - 15)^2 + 10$

$Solve\ f'(x) = g'(x)\ for\ x$

$x = -\frac{15b}{a-b}$

Substitute this expression for x into $f(x) = g(x)$ and solve for a.

$a = \frac{2}{45} - \frac{4}{45(45b+2)}$

$= \frac{2b}{45b+2}$

1 mark for correctly equating equations.
1 mark for a valid process to determine required relationship from the equations.
1 mark for a correct equation for a in terms of b.

(iv) If $b = -\frac{1}{5}$, construct a hybrid function, called $p(x)$, that models the profile of the roof from points O to E, inclusive. Include relevant subdomains and any parameter values. **3 marks**

$a = \frac{2b}{45b+2} | b = -\frac{1}{5}$

$= \frac{2}{35}$

$f(x) = \frac{2}{35}x^2$

$g(x) = -\frac{1}{5}(x - 15)^2 + 10$

Determine x-coordinate of the point of intersection of $f(x)$ and $g(x)$ by solving $f(x) = g(x)$ for x. Alternatively, substitute the values of a and b into the expression for x determined in **Part (iii)**.

$x = \frac{35}{3}$

Let $h(x)$ represent the profile of the last section.

$h(x) = \frac{2}{35}(x - 30)^2$

By symmetry, it will have the same dilation factor as $f(x)$ and be translated 30 units right. The x-coordinate of the point of intersection of $g(x)$ and $h(x)$ can also be determined by symmetry (or by solving $g(x) = h(x)$ for x

$$x = 15 + \left(15 - \frac{35}{3}\right)$$

$$= \frac{55}{3}$$

$p(x) = \{\frac{2}{35}x^2, \quad 0 \le x \le \frac{35}{3} \quad -\frac{1}{5}(x - 15)^2 + 10, \quad \frac{35}{3} < x \le \frac{55}{3} \quad \frac{2}{35}(x - 30)^2, \quad \frac{55}{3} < x \le 30$

1 mark for correct value of a.
1 mark for correct x-coordinates of connection points.
1 mark for correct hybrid function with valid, non-overlapping, subdomains.

Question 4 (14 marks)

a. The time required for a group to finish their booking at a particular restaurant follows a normal distribution with a mean of 45 minutes and a standard deviation of 10 minutes. Round all probabilities to 4 decimal places.

(i) Determine the probability, that a group will finish their booking in less than 30 minutes. **1 mark**

Let X be the RV representing the time required for a group to finish their booking.

Use Normal Cdf on CAS.

$\Pr Pr\,(X < 30) = 0.0668$

1 mark for correct probability rounded to 4 decimal places.

(ii) Determine the probability that of the four groups about to start their booking, at least three will take more than 60 minutes. **2 marks**

Use symmetry from **Part (i)**, Normal Cdf on CAS.

$\Pr Pr\ (X > 60) = 0.0668$

A group will either take longer than 60 minutes, or it won't. Therefore Binomial distribution problem.

Let Y be the RV representing number of groups that take more than 60 minutes.

$\Pr Pr\ (p = 0.0668,\ n = 4) = 0.0011$ to 4 decimal places.

1 mark for correct value for $\Pr Pr\ (X > 60)$.
1 mark for correct value for $\Pr Pr\ (p = 0.0668,\ n = 4)$.

(iii) When a group finishes a booking at a particular table, it takes 5 minutes to prepare the table for the next booking. The restaurant is open for 4 hours. Determine the probability that 6 groups will be able to complete their booking at a particular table. **3 marks**

For 6 sequential groups at a table, the number of clean ups will be 5 (between the first group and second, second group and third and so on). Each group follows the same distribution regarding time required to finish booking.

$E(6X + (5\times5)) = E(6X + 25)$

$= 6E(X) + 25$

$= 295\ minutes$

$VAR(6X + (5\times5)) = VAR(6X)$

$= 6^2 VAR(X)$

$= 36\times10^2$

$= 3600$

$\therefore SD(X) = 36\ minutes$

$\therefore \Pr Pr\ (6X + 25|\mu = 295,\ \sigma = 36) < 4\times60 \quad = 0.0633$

1 mark for correct mean.
1 mark for correct standard deviation.
1 mark for correct probability rounded to 4 decimal places.

b. **The waiters of the restaurant receive tips for their service. The total amount of tips, X (dollars), a particular waiter receives in an evening follows a probability density function given by:**

$$f(x) = \{ax^{\frac{1}{3}} \quad 0 \le x \le c\ 0 \qquad elsewhere$$

The probability that the waiter receives at most \$50 dollars in tips in an evening is 0.4605. The probability that the waiter receives between \$30 and \$70 in tips in an evening is 0.4882.

(i) **Determine the value of the parameter a to 6 decimal places, and hence determine the maximum amount the waiter can receive in tips in an evening, rounded to the nearest dollar.** **2 marks**

$$\Pr Pr\,(X \le 50) \ = \int_0^{50} ax^{\frac{1}{3}} dx$$

$$0.4605 = \int_0^{50} ax^{\frac{1}{3}}\, dx$$

$$a = 0.003333$$

Alternatively:

$$\Pr Pr\,(30 \le X \le 70) \ = \int_{30}^{70} ax^{\frac{1}{3}} dx$$

$$0.4882 = \int_{30}^{70} ax^{\frac{1}{3}}\, dx$$

$$a = 0.003333$$

Maximum value is equal to *c*.

$$solve \int_o^c 0.003333x^{1/3} dx = 1 \ for\ c$$

Maximum value is \$89 rounded to the nearest dollar.

1 mark for the correct value of a rounded to 6 decimal places.
1 mark for correct maximum value rounded to the nearest dollar.

(ii) One evening the waiter received an amount in tips equal to the mean value, while on a different evening the amount received was equal to the median value. What is the magnitude of the difference between what was received on these evenings? Round the answer to the nearest 10 c. 3 marks

Determine the value of c.

$$solve \int_0^c f(x)dx = 1$$

$c = 89.44$

Determine expected value of *X*.

$$E(X) = \int_0^{89.45} xf(x)\,dx$$

$$= \$51.11$$

Determine median value of *X*.

$$Solve \int_0^m f(x)dx = \frac{1}{2}$$

$$= \$53.19$$

$$Mean - Median = 51.11 - 53.19$$

= -$2.08

The magnitude of the difference between the mean and median is $2.1 rounded to the nearest 10c.

1 mark for correct median value.
1 mark for correct mean value.
1 mark for correct magnitude of the difference rounded to the nearest 10c.

c. The owner of the restaurant wanted to determine the level of customer satisfaction with the menu. Of the 16 people questioned in a preliminary survey, 11 indicated that they were satisfied with the menu.

(i) What is the value of $\hat{p}$? 1 mark

$$\hat{p} = \frac{11}{16}$$

1 mark for the correct value of $\hat{p}$.

(ii) The owner wanted to construct a 95% confidence interval of width 0.10. How many people should be surveyed for this condition to be achieved? Round your answer to the nearest 10 people. **2 marks**

For a 95% C.I, $Z = 1.96$

$$\therefore C.I = \left(\hat{p} - Z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + Z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

$$= \left(\frac{11}{16} - 1.96\sqrt{\frac{\frac{11}{16}\left(1-\frac{11}{16}\right)}{n}}, \frac{11}{16} + 1.96\sqrt{\frac{\frac{11}{16}\left(1-\frac{11}{16}\right)}{n}}\right)$$

$$= \left(0.6875 - \frac{0.9085}{\sqrt{n}}, 0.6875 + \frac{0.9085}{\sqrt{n}}\right)$$

If the interval width is 0.10 then:

$$0.10 = \left(0.6875 + \frac{0.9085}{\sqrt{n}}\right) - \left(0.6875 - \frac{0.9085}{\sqrt{n}}\right)$$

Solve for n.

$n = 330$ people to the nearest 10 people.

1 mark for the correct confidence interval in terms of n.
1 mark for the correct number of people rounded to the nearest 10 people.

Question 5 (7 marks)

a. What is the purpose of the variable c? 1 mark

To count the number of iterations of the algorithm.

1 mark for a valid explanation of the variable c.

b. Perform a desk check by completing the table below (provide exact value or round to 3 decimal places for non-terminating decimals). 3 marks

c	m	n	$n-m$	b	$f(b)$
1	1	2	1	1.5	1.107
2	1.5	2	0.5	1.75	0.395
3	1.75	2	0.25	1.875	-0.071

1 mark for correct values in the first row.
1 mark for correct values in the second row.
1 mark for correct values in the third row.

c. What will be the value of c when the condition to end the while command is satisfied? 1 mark

Note that $n - m$ is halving with each iteration. Use a table or repeated halving of the CAS to determine at which iteration the condition is first satisfied. Alternatively:

$$m - n = \left(\frac{1}{2}\right)^{c-1}$$

Solve the above equation for $c \, for \, m - n = 0.001$.

$$c = 9.97$$

It will therefore take 10 iterations of the algorithm for the condition to the satisfied, hence $c = 10$.

1 mark for the correct value of c.

d. Modify the pseudocode, by completing the table below, so that it approximates the other point of intersection of $p(x)$ and $q(x)$ and that the accuracy of the approximation is increased by a factor of ten.

If a line of the code is unchanged, write the letter U in the adjacent cell. 2 marks

From their graphs, $p(x)$ intersects $q(x)$ between $x = 4$ and $x = 5$.

Original Pseudocode	Modified Pseudocode
define $f(x)$:	U
return $e^x - x^3$	U
$m \leftarrow 1$	$m \leftarrow 4$
$n \leftarrow 2$	$n \leftarrow 5$
$c \leftarrow 1$	U
while $n - m > 0.001$	while $n - m > 0.0001$
$b \leftarrow \frac{n+m}{2}$	U
if $f(m) \times f(b) < 0$ then	U
$n \leftarrow b$	U
Else	U
$m \leftarrow b$	U
end if	U
$c \leftarrow c + 1$	U
end while	U
print $b,\ f(b),\ c$	U

1 mark for a valid set of values for m and n.
1 mark for correct adjustment to 'while $n - m > 0.001$'.

