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Student Name……………………………………

MATHEMATICAL METHODS UNITS 3 & 4

TRIAL EXAMINATION 2

2024

Reading Time: 15 minutes
Writing time: 2 hours

Instructions to students

This exam consists of Section A and Section B.
Section A consists of 20 multiple-choice questions, which should be answered on the detachable answer sheet which can be found on page 25 of this exam.
Section B consists of 5 extended-answer questions.
Section A begins on page 2 of this exam and is worth 20 marks.
Section B begins on page 9 of this exam and is worth 60 marks.
There is a total of 80 marks available.
All questions in Section A and Section B should be answered.
In Section B, where more than one mark is allocated to a question, appropriate working must be shown.
Where a numerical answer is required, an exact value must be given unless otherwise directed.
Diagrams in this exam are not to scale except where otherwise stated.
Students may bring one bound reference into the exam.
Students may bring into the exam one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory does not need to be cleared. For approved computer-based CAS, full functionality may be used.
A formula sheet can be found on the last page of this exam.

SECTION A – Multiple-choice questions

Question 1

The amplitude and the period of the function $g(x) = -2\cos(4x - \pi)$, are given respectively by

A. -2 and $-\frac{\pi}{2}$

B. -2 and $\frac{\pi}{2}$

C. 2 and $-\frac{\pi}{2}$

D. 2 and $\frac{\pi}{2}$

Question 2

The function g is continuous over its domain $(-6, 1]$, and the function h is continuous over its domain $[-3, 2)$. Let $f(x) = g(x) \times h(x)$.
The domain of f is

A. $(-6, 2)$

B. $(-6, -3]$

C. $[-3, 1]$

D. $(-3, 1)$

Question 3

The equation $x^2 - 6x + k = 0$ has two real solutions if

A. $k \in (-\infty, 9)$

B. $k \in (-\infty, 9]$

C. $k \in \{-9, 9\}$

D. $k \in (9, \infty)$

Question 4

The graph of the function $f(x) = x^{\frac{2}{3}} - 2$ has a positive x-intercept at $x = a$.

Newton's method is used to find the approximate value of a, with an initial estimate of $x_0 = 2$ being used.
If x_n correctly approximates the value of a to four decimal places, then the minimum value of n is

A. 1

B. 2

C. 3

D. 4

Question 5

A system of simultaneous linear equations is given by

$$6x + 2ay = 3$$
$$3ax + y = a$$

There will be a unique solution only when

A. $a = -1$ or $a = 1$

B. $a = -2$ or $a = 3$

C. $a \in R \setminus \{-2, 2\}$

D. $a \in R \setminus \{-1, 1\}$

Question 6

Let $f: [-a, a] \rightarrow R, f(x) = \sin\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right)$.
If the inverse function f^{-1} exists, then the maximum value of a is

A. $\frac{\pi}{4}$

B. $\frac{3\pi}{4}$

C. π

D. $\frac{5\pi}{4}$

Question 7

Let $f(x) = x - 1$ and $g(x) = 4 - x^2$.

The composite function h is given by $h:[-2, 3) \to R$, $h(x) = (g \circ f)(x)$.

The range of h is

A. $[-5, 0)$

B. $(-5, 0]$

C. $[-5, 4)$

D. $[-5, 4]$

Question 8

Consider the function f given by

$$f(x) = \begin{cases} e^{ax} & -\infty < x \leq 0 \\ b - e^{-x} & 0 < x < \infty \end{cases}$$

The graph of $y = f(x)$ is continuous and smooth at $x = 0$ when

A. $a = -1$ and $b = -2$

B. $a = -1$ and $b = 2$

C. $a = 1$ and $b = -2$

D. $a = 1$ and $b = 2$

Question 9

If $\int_2^5 f(x)dx = 1$ and $\int_3^5 f(x) = -2$ then $\int_2^3 (f(x) - 2x)dx$ is equal to

A. -7

B. -6

C. -2

D. 8

Question 10

The two functions f and g are continuous and differentiable for $x \in R$, where $f(4) = 3$, $g(4) = -1$, $f'(4) = -2$ and $g'(4) = 5$.

At the point where $x = 4$, the gradient of the graph of $y = \dfrac{f(x)}{g(x)}$ is

A. -17

B. -13

C. -3

D. $-\dfrac{2}{5}$

Question 11

A body corporate surveyed 90 randomly selected residents of a very large apartment complex and found that one-sixth of the surveyed residents owned a pet.
The body corporate calculated a $p\%$ confidence interval of $(0.1163, 0.2170)$ correct to four decimal places.

The value of p is closest to

A. 75
B. 80
C. 90
D. 95

Question 12

Let A and B be two independent events from a sample space where $\Pr(A) = 3\Pr(B)$ and $\Pr(A \cup B) = 0.37$.

$\Pr(A|B)$ is equal to

A. 0.1
B. 0.2
C. 0.3
D. 0.4

Question 13

The weights of fresh lemons picked from a particular lemon orchard are normally distributed.
If 10% of the lemons weigh more than 66 grams and 15% of the lemons weigh less than 56 grams, the mean weight, in grams, of a fresh lemon from the orchard is closest to

A. 60.0
B. 60.5
C. 61.0
D. 61.5

Question 14

The following algorithm uses the trapezium rule to estimate the area under a graph over a particular interval.

```
Inputs:     f(x), the function describing the graph
            a, the left endpoint of the interval
            b, the right endpoint of the interval
            n, the number of trapeziums used
Define:     trapezium (f(x),a,b,n)
            h ⟵ (b - a) ÷ n
            total ⟵ f(a) + f(b)
            x ⟵ a + h
            i ⟵ 1
            While i < n  Do
                total ⟵ total + 2 × f(x)
                x ⟵ x + h
                i ⟵ i + 1
            EndWhile
            area estimate ⟵ total × (h ÷ 2)
            Print  area estimate
```

The algorithm is run using the instruction

trapezium ($x^2 + 1$, 0, 3, 3)

The output is

A. $\frac{15}{2}$

B. $\frac{25}{2}$

C. $\frac{15}{2}, \frac{25}{2}$

D. $\frac{15}{2}, \frac{25}{2}, 20$

Question 15

A purse contains four gold coins and five silver coins. Four coins are randomly drawn from the purse without replacement.
The probability that an equal number of gold and silver coins are drawn, given that at least one silver coin is drawn, is

A. $\frac{1}{125}$

B. $\frac{2}{25}$

C. $\frac{12}{25}$

D. $\frac{5}{9}$

Question 16

A fair sided die is being rolled. For random samples of n die rolls, where n is an integer greater than two, $\hat{P}$ is the random variable that represents the proportion of die rolls that have a number less than three thrown.
If $\Pr\left(\hat{P} > \frac{1}{n}\right) \geq 0.85$, the smallest value of n is

A. 6
B. 7
C. 8
D. 9

Question 17

The tangent to the graph of $y = 2\tan\left(\frac{x}{2}\right) + 3$ is parallel to the line with equation $y = 2x - 5$ at particular points on the graph.

Which one of the following gives all the possible values of x at these points?

A. $2k\pi \pm \frac{\pi}{2},\ k \in Z$

B. $-\frac{\pi}{4}, \frac{\pi}{4}$

C. $4k\pi \pm \frac{\pi}{3},\ k \in Z$

D. $\frac{(8k \pm 1)\pi}{2},\ k \in Z$

Question 18

Let $f:(0, 5\pi] \to R,\ f(x) = \sin(x)\log_e(x)$.
The graph of $y = f(x-k)$, where $k \in R$, has two or more stationary points with positive x-coordinates.
Which one of the following is **not** a possible value of k?

A. -4π

B. -3π

C. 1

D. 2π

Question 19

The probability density functions of two normal random variables X and Y are $f(x)$ and $g(x)$ respectively.
The means of X and Y are both zero and the standard deviation of X is half the standard deviation of Y.

A transformation maps the graph of f onto the graph of g, where $g(x) = a f\left(\frac{x}{b}\right)$.

The values of a and b could be

A. $a = -\frac{1}{2}$ and $b = 2$

B. $a = \frac{1}{2}$ and $b = 2$

C. $a = 2$ and $b = -\frac{1}{2}$

D. $a = 2$ and $b = \frac{1}{2}$

Question 20

Let $f(x) = \sqrt{ax+b}$ and let g be the inverse function of f.
Given that $f(0) = 2$ and $g'(x) < 0$, then all the possible values of a are given by

A. $a \in R^-$

B. $a \in [0, 2)$

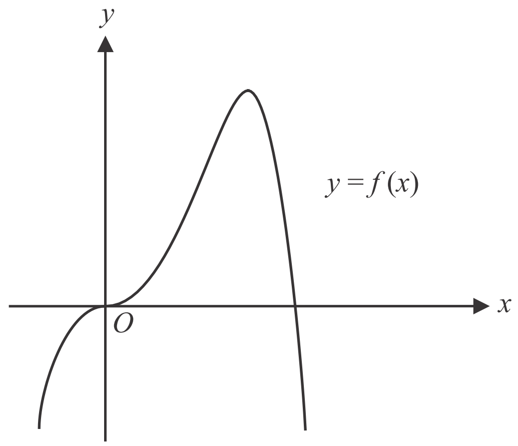
C. $a \in R\backslash\{0\}$

D. $a \in R^+$

SECTION B

Question 1 (10 marks)

The diagram below shows part of the graph of the quartic function $f: R \to R, f(x) = -\frac{x^4}{3} + \frac{4x^3}{3}$.



a. Find the coordinates of the point at which the maximum value of f occurs. 1 mark

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b. State the nature of the stationary point on the graph of f at $x = 0$. 1 mark

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c. Find the values of x for which $f'(x) > 0$. 1 mark

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d. Find the values of h such that $f(x+h)=0$ has no positive solutions. 1 mark

e. Find the values of k such that $f(x)+k=0$ has no solutions. 1 mark

f. Find the equation of the tangent to f at $x=-1$. 1 mark

g. The graph of $y=f(x)$ has a tangent at the point where $x=q$, $q \neq -1$, that is parallel to the tangent found in **part f**. Find the value of q. 1 mark

h. A straight line passes through the origin and is perpendicular to the tangent found in **part f**. This straight line intersects the tangent at the point A.

It also intersects the tangent to f at $x=q$ at the point $B\left(\frac{256}{265}, -\frac{48}{265}\right)$.

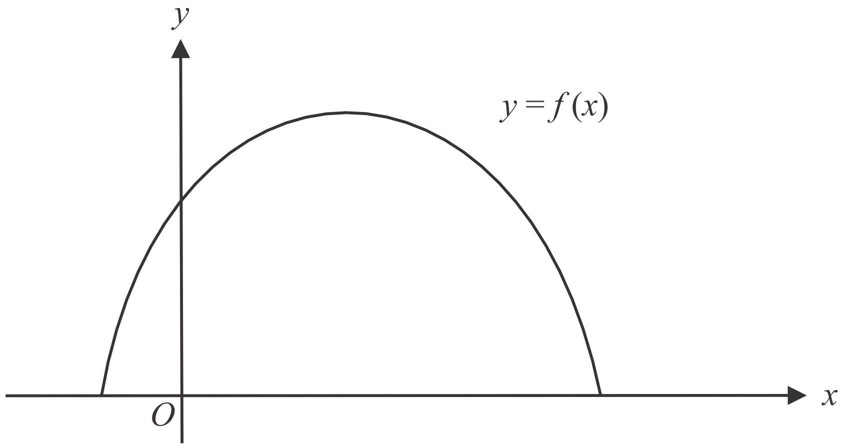
Find the midpoint of the line segment AB. 3 marks

Question 2 (12 marks)

The parabolic cross-section of a rail tunnel is modelled by the function

$$f(x) = -\frac{2}{9}(x+a)(x-5a),\ a>0,\ f(x)\geq 0$$

where x is the horizontal distance, in metres, from an origin O and y is the vertical distance, in metres, above the horizontal base of the tunnel. The graph of f is shown below.



a. **i.** Write down, in terms of a, the domain of f. 1 mark

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ii. State the equation of the axis of symmetry of the graph of f in terms of a. 1 mark

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iii. Find the values of x, in terms of a, for which f is strictly increasing. 1 mark

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b. Find the values of a for which the maximum height of the tunnel is less than twice the width of the base of the tunnel. 2 marks

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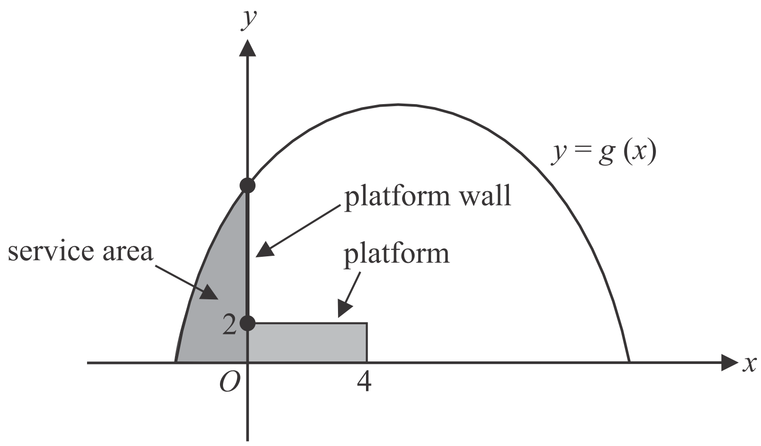
The diagram below shows the parabolic cross-section of the rail tunnel at a particular station.



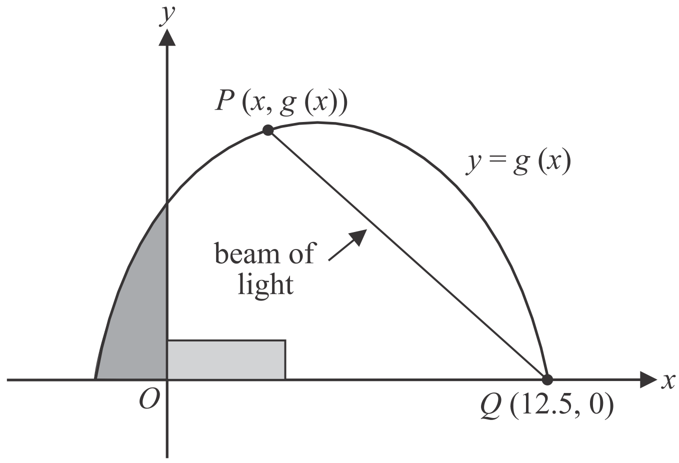
The rectangular cross-section of the platform has a width of four metres and a height of two metres.
The vertical platform wall lies on the y-axis.

The cross-section of a service area that lies behind the vertical platform wall is also shown.
At this particular station, the cross-section of the rail tunnel is modelled by the function

$$g(x) = -\frac{2}{9}(x+2.5)(x-12.5), \quad g(x) \geq 0.$$

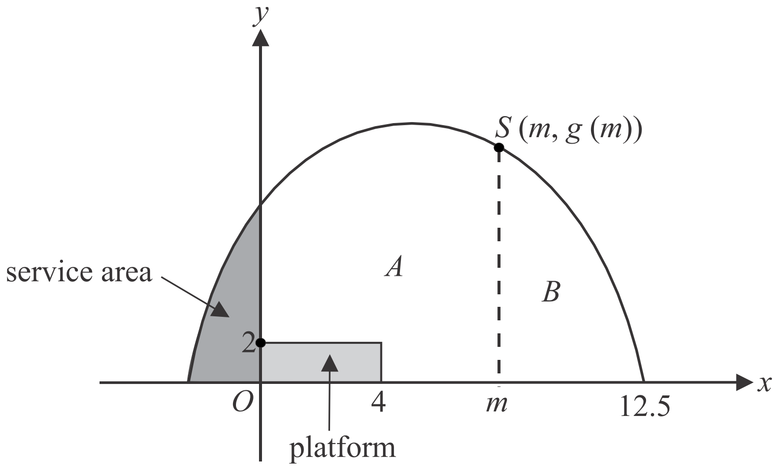
c. Find the vertical distance, in metres, from the platform to the top of the vertical platform wall. 1 mark

During construction of this station, a spotlight is installed on the roof of the tunnel at the point $P(x, g(x))$ where $x \geq 0$. This spotlight must be able to project a straight beam of light from P to the point $Q(12.5, 0)$ as indicated in the diagram below.



d. Find the maximum length of the beam of light that the spotlight must be able to project. Give your answer, in metres, correct to one decimal place. 3 marks

A water sprinkler is to be attached to the roof of the tunnel at this station at the point $S(m, g(m))$.



Let the cross-sectional area of the tunnel to the left of the line $x = m$, excluding the cross-sectional areas of the platform and the service area, be area A.
Let the cross-sectional area of the tunnel to the right of the line $x = m$, be area B.

e. Find the value of m if area A is equal to area B. Give your answer correct to two decimal places. 3 marks

Question 3 (15 marks)

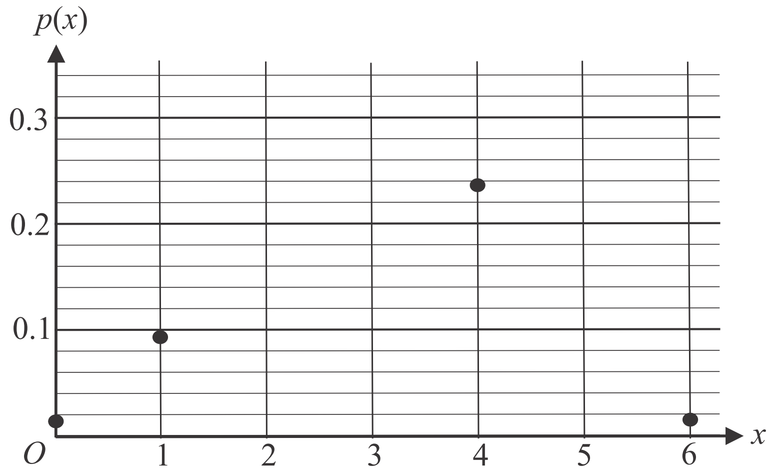
An online food delivery company delivers ready-made meals to customers' homes. Due to a decline in business revenue, a manager is reviewing the company's service history.
It is found that the number of minutes before or after the time the food is expected to have arrived at the customer's address, is a normally distributed variable, with a mean of zero minutes and a standard deviation of five minutes.

a. Find the probability that a meal arrives within two minutes of its expected arrival time. Give your answer correct to four decimal places. 1 mark

Customers receive a refund if their meal arrives more than *k* minutes after the expected arrival time.

b. If five percent of customers are refunded, find value of *k*, correct to three decimal places. 1 mark

To analyse the performance of the delivery drivers, the manager models the situation where six deliveries are made and the probability that a delivery is late is 0.5.
He makes the assumption that the arrival time of a delivery is independent of the arrival time of any other delivery.
The graph below shows an incomplete plot of the probability mass function, $p(x)$, for the number of times, x, that a meal is delivered late across six deliveries.



c. **i.** Complete the graph above. 2 marks

(*Answer on the graph above.)*

ii. Find the probability that at least one of the deliveries is delivered late.
Give your answer correct to four decimal places. 1 mark

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iii. Find the probability that no more than four deliveries are delivered late given that at least one of the deliveries is late.
Give your answer correct to four decimal places. 2 marks

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Unfortunately, not all deliveries are delivered to the correct address. For a particular driver, on any given day, there are at most two meals that he delivers to the incorrect address.
The number of meals that he delivers to the incorrect address is a random variable X which has the following discrete probability distribution.

x	0	1	2
$\Pr(X = x)$	0.8	m	n

d. **i.** Find the variance of the number of meals delivered to the incorrect address in terms of m.
Express your answer in the form $am^2 + bm + c$ where a, b and c are real constants. 3 marks

ii. Hence, find the maximum variance of the number of meals delivered to the incorrect address. 1 mark

For random samples of 1000 delivered meals, $\hat{P}$ is the random variable that represents the proportion of meals that are delivered to the incorrect address.

The food delivery company analysed a sample of 1000 delivered meals and calculated a 95% confidence interval for the proportion of meals delivered to the incorrect address. The confidence interval they obtained was (0.1752, 0.2248).

e. **i.** Find the estimated standard deviation of $\hat{P}$.
Give your answer correct to four decimal places. 2 marks

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ii. Find the least number of delivered meals required to be sampled to reduce the width of this confidence interval by at least a quarter. 2 marks

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Question 4 (13 marks)

Consider the functions $f:[0,\infty) \to R, f(x) = \log_e(x+3)$ and $g:D \to R,\ g(x) = \log_e(3-x)$ where D is the maximal domain of g.

a. State D using interval notation. 1 mark

b. Find the area bounded by the graphs of f and g and the line $x = 2$. 2 marks

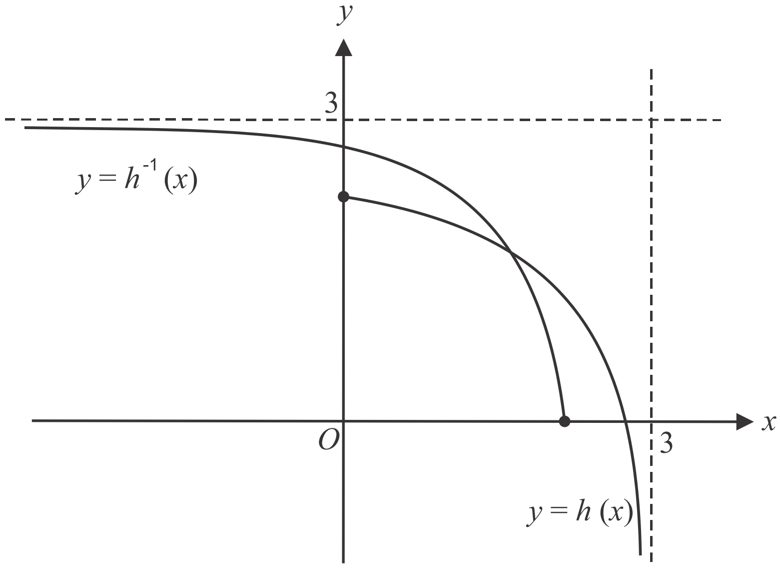
Let $h(x) = f(x) + g(x)$ where the function h is defined.

c. **i.** Show that $h(x) = \log_e(9 - x^2)$. 1 mark

ii. Find the maximal domain and range of h. 2 marks

iii. Find the rule for h^{-1}, the inverse function of h. 1 mark

Part of the graphs of h and h^{-1} over their maximal domains are shown below.



Let p be the x-coordinate of the point of intersection of the graphs of h and h^{-1}.

d. Write down an expression that includes p and the function $h^{-1}(x)$ and which gives the average value of the function $h(x)$ over the interval $[0, p]$. Do not evaluate this expression. 2 marks

Consider the graph of the function $j_a : (-a, a) \rightarrow R,\ j_a(x) = \log_e\left(a^2 - x^2\right)$ where $a > 1$.

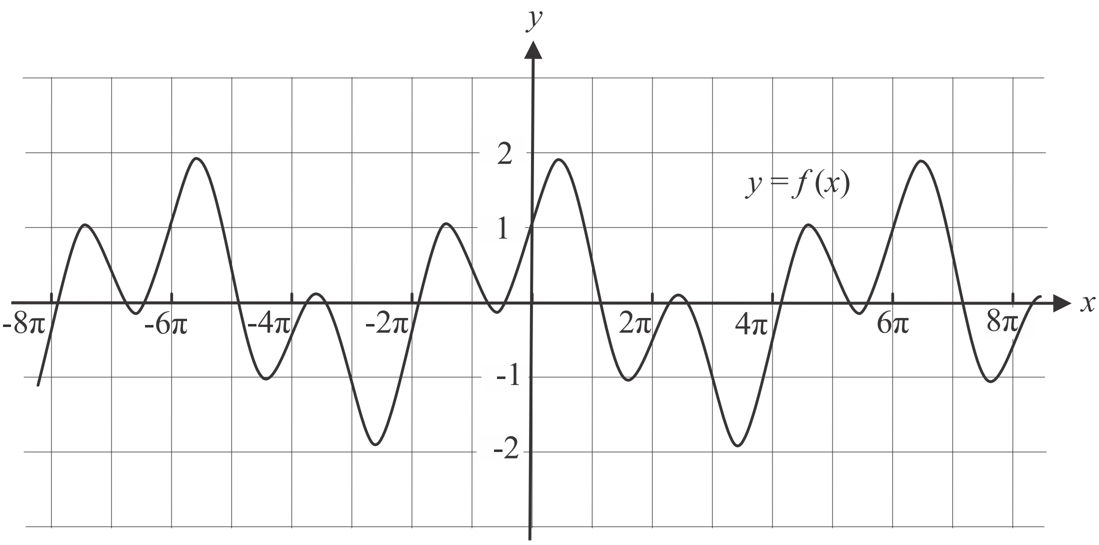
Let Q be a point on j_a where $j_a(x) = 0$ and $x < 0$.

e. Find, in terms of *a,* the gradient of the tangent to the graph of $y = j_a(x)$ at the point Q. 2 marks

f. Find the values of a for which the angle that the tangent to $j_a(x)$ at the point Q makes with the positive branch of the x-axis is less than 45°. 2 marks

Question 5 (10 marks)

Let $f : R \to R,\ f(x) = \sin(x) + \cos\left(\frac{x}{3}\right)$. Part of the graph of *f* is shown below.



a. State the period of *f*. 1 mark

b. State the maximum value of *f*, correct to three decimal places. 1 mark

c. Find the smallest positive value of *c* for which $f(x) = -f(x+c)$. 1 mark

d. Find the smallest value of q, where $q > 0$, for which the area bounded by f and the x-axis over the interval $\left[0, q\right]$, is equal above and below the x-axis. Give your answer correct to two decimal places. 2 marks

Consider the functions of the form $f_a : R \to R$, $f_a(x) = \sin(ax) + \cos\left(\frac{x}{3a}\right)$, where a is a positive integer.

e. Find, in terms of a, the period of the functions f_a. 1 mark

f. Find the values of x, in terms of a, for which the maximum value of the following functions occur.

i. $y = \sin(ax)$ 1 mark

ii. $y = \cos\left(\frac{x}{3a}\right)$ 1 mark

g. Hence show that the maximum value of f_a can never equal two. 2 marks

MATHEMATICAL METHODS

TRIAL EXAMINATION 2

MULTIPLE-CHOICE ANSWER SHEET

STUDENT NAME: ________________________________

INSTRUCTIONS

Fill in the letter that corresponds to your choice. Example: A ▬ C D

The answer selected is B. Only one answer should be selected.

1. A B C D
2. A B C D
3. A B C D
4. A B C D
5. A B C D
6. A B C D
7. A B C D
8. A B C D
9. A B C D
10. A B C D
11. A B C D
12. A B C D
13. A B C D
14. A B C D
15. A B C D
16. A B C D
17. A B C D
18. A B C D
19. A B C D
20. A B C D

Mathematical Methods formula sheet

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$		$\int x^n dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right)=an(ax+b)^{n-1}$		$\int (ax+b)^n dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,\ n\neq -1$	
$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$		$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$	
$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$		$\int \frac{1}{x}dx=\log_e(x)+c,\ x>0$	
$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$		$\int \sin(ax)dx=-\frac{1}{a}\cos(ax)+c$	
$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$		$\int \cos(ax)dx=\frac{1}{a}\sin(ax)+c$	
$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}=a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$	Newton's method	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$
trapezium rule approximation	$Area\approx\frac{x_n-x_0}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-2})+2f(x_{n-1})+f(x_n)\right]$		

Probability

$\Pr(A)=1-\Pr(A')$		$\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A\cap B)$	
$\Pr(A\mid B)=\frac{\Pr(A\cap B)}{\Pr(B)}$			
mean	$\mu=\mathrm{E}(X)$	variance	$\mathrm{var}(X)=\sigma^2=\mathrm{E}((X-\mu)^2)=\mathrm{E}(X^2)-\mu^2$
binomial coefficient	$\binom{n}{x}=\frac{n!}{x!(n-x)!}$		

Probability distribution		Mean	Variance
discrete	$Pr(X=x)=p(x)$	$\mu=\Sigma x\,p(x)$	$\sigma^2=\Sigma(x-\mu)^2\,p(x)$
binomial	$\Pr(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$	$\mu=np$	$\sigma^2=np(1-p)$
continuous	$\Pr(a<X<b)=\int_a^b f(x)dx$	$\mu=\int_{-\infty}^{\infty}x\,f(x)\,dx$	$\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$

Sample proportions

$\hat{P}=\frac{X}{n}$		mean	$\mathrm{E}(\hat{P})=p$
standard deviation	$\mathrm{sd}(\hat{P})=\sqrt{\frac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$

End of formula sheet