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MATHS METHODS UNITS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2024

Question 1 (3 marks)

a. $y = \cos\left(1 - x^2\right)$

$$\begin{aligned}\frac{dy}{dx} &= -\sin\left(1 - x^2\right) \times -2x \qquad \text{(chain rule)}\\ &= 2x\sin\left(1 - x^2\right)\end{aligned}$$

(1 mark)

b. $f(x) = \dfrac{\sin(2x)}{1 + e^{2x}}$

$$f'(x) = \frac{\left(1 + e^{2x}\right) \times 2\cos(2x) - 2e^{2x} \times \sin(2x)}{\left(1 + e^{2x}\right)^2} \qquad \text{(quotient rule)}$$

(1 mark)

$$\begin{aligned}f'(0) &= \frac{\left(1 + e^0\right) \times 2\cos(0) - 2e^0 \times \sin(0)}{\left(1 + e^0\right)^2}\\ &= \frac{(1+1) \times 2 \times 1 - 2 \times 1 \times 0}{(1+1)^2}\\ &= \frac{4-0}{2^2}\\ &= 1\end{aligned}$$

(1 mark)

Question 2 (3 marks)

a. $(f \circ g)(x) = \log_e\left(x^2 + 1\right)$ **(1 mark)**

b. $d_{f \circ g} = d_g = R$ **(1 mark)**

$r_{f \circ g} = [0, \infty)$ **(1 mark)**

Note that the minimum value of x^2 is zero, so the minimum value of $x^2 + 1$ is 1, and $\log_e(1) = 0$.

Question 3 (5 marks)

a. average rate of change $= \dfrac{f\left(\frac{\pi}{8}\right)-f(0)}{\frac{\pi}{8}-0}$

$= \dfrac{2-1}{\frac{\pi}{8}}$

$= \dfrac{8}{\pi}$ **(1 mark)**

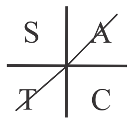
Note that the value of $f\left(\frac{\pi}{8}\right)$ can be found using the symmetry of the graph, i.e. the graph passes through the point $\left(\frac{\pi}{8}, 2\right)$.

Alternatively, it can be found by evaluating $\tan\left(\frac{\pi}{4}\right)+1=2$.

b. $f(x) = 1+\dfrac{1}{\sqrt{3}}$

$\tan(2x)+1 = 1+\dfrac{1}{\sqrt{3}}$

$\tan(2x) = \dfrac{1}{\sqrt{3}}$

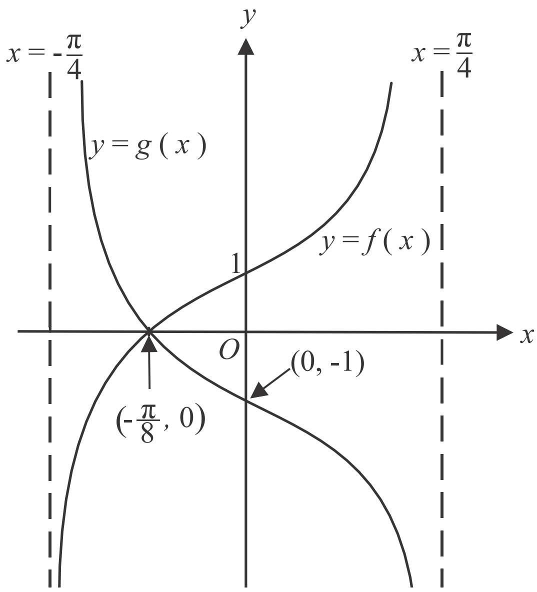


$2x = \ldots -\dfrac{5\pi}{6}, \dfrac{\pi}{6}, \dfrac{7\pi}{6}, \ldots$ base angle $= \dfrac{\pi}{6}$ **(1 mark)**

$x = \ldots -\dfrac{5\pi}{12}, \dfrac{\pi}{12}, \dfrac{7\pi}{12}, \ldots$

But $d_f = \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and from the graph, we see that there would only be one point of intersection between the graph of *f* and the line $y = 1+\dfrac{1}{\sqrt{3}}$. So $x = \dfrac{\pi}{12}$. **(1 mark)**

c. The graph of *f* is reflected in the *y*-axis and then translated 2 units down to obtain the graph of *g*.



(1 mark) - correct shape
(1 mark) – correct axis intercepts

Question 4 (4 marks)

a. $\int_0^{e-1} \frac{3}{x+1} dx = \left[3\log_e(x+1)\right]_0^{e-1}$ **(1 mark)**

$= 3\left(\log_e(e-1+1) - \log_e(1)\right)$

$= 3\left(\log_e(e) - 0\right)$

$= 3 \times 1$

$= 3$ **(1 mark)**

b. $f'(x) = 2\sin(\pi x)$

$f(x) = \int 2\sin(\pi x)dx$

$= -\frac{2}{\pi}\cos(\pi x) + c$ **(1 mark)**

Since $f\left(\frac{1}{3}\right) = 0$,

$0 = -\frac{2}{\pi}\cos\left(\frac{\pi}{3}\right) + c$

$c = \frac{2}{\pi} \times \frac{1}{2}$

$= \frac{1}{\pi}$

So $f(x) = -\frac{2}{\pi}\cos(\pi x) + \frac{1}{\pi}$. **(1 mark)**

Question 5 (4 marks)

a.

$$y = x^2\log_e(2x)$$

$$\frac{dy}{dx} = 2x\log_e(2x) + x^2 \times \frac{2}{2x}$$

$$= 2x\log_e(2x) + x$$

(1 mark)

b.

$$\text{average value} = \frac{1}{1-\frac{1}{2}}\int_{\frac{1}{2}}^{1} f(x)\,dx$$

$$= 2\int_{\frac{1}{2}}^{1} x\log_e(2x)\,dx$$

$$= \int_{\frac{1}{2}}^{1} 2x\log_e(2x)\,dx$$ **(1 mark)**

From part **a.** $\frac{dy}{dx} = 2x\log_e(2x) + x$

$$\int\left(2x\log_e(2x) + x\right)dx = x^2\log_e(2x)$$

$$\int 2x\log_e(2x)\,dx + \int x\,dx = x^2\log_e(2x)$$

$$\int 2x\log_e(2x)\,dx = x^2\log_e(2x) - \int x\,dx$$

$$= x^2\log_e(2x) - \frac{x^2}{2} + c$$

So average value $= \left[x^2\log_e(2x) - \frac{x^2}{2}\right]_{\frac{1}{2}}^{1}$ **(1 mark)**

$$= \left(\log_e(2) - \frac{1}{2}\right) - \left(\frac{1}{4}\log_e(1) - \frac{1}{8}\right)$$

$$= \log_e(2) - \frac{1}{2} - 0 + \frac{1}{8}$$

$$= \log_e(2) - \frac{3}{8}$$

$$= \log_e(2) - \frac{3}{2^3}$$ **(1 mark)**

Question 6 (2 marks)

$E(\hat{P}) = p$

$\Pr(\hat{P}=0) = \binom{4}{0} p^0 (1-p)^4 = \frac{1}{4}$ **(1 mark)**

$$1-p = \pm \frac{1}{\sqrt{2}}$$

Since $0 < p < 1$, then $p = 1 - \frac{1}{\sqrt{2}}$. **(1 mark)**

Question 7 (4 marks)

a. Stationary points occur when $f'(x) = 0$

$$f(x) = x + \frac{1}{x-2}$$

$$= x + (x-2)^{-1}$$

$$f'(x) = 1 - (x-2)^{-2}$$

We require that $1 - \frac{1}{(x-2)^2} = 0$

$$1 = \frac{1}{(x-2)^2}$$

$$(x-2)^2 = 1$$

$$x - 2 = \pm 1$$

$$x = 1 + 2 \quad \text{or} \quad x = -1 + 2$$

$x = 3$ or $x = 1$ **(1 mark)**

$$f(1) = 1 + \frac{1}{-1} = 0 \quad \text{and} \quad f(3) = 3 + \frac{1}{3-2} = 4$$

The stationary points are (1, 0) and (3, 4). **(1 mark)**

b. i. Looking at the stationary points on the graph and using our answers to part **a.**, we note that if the graph of *f* is translated by **between** 0 and 4 units downwards, then the graph of *f* will not intersect with the *x*-axis. Hence there will be no solutions to the equation $f(x) + c = 0$.
So we require that $-4 < c < 0$. **(1 mark)**

ii. Method 1

In order to become the graph of $y = 1 + f(a - x)$, the graph of $y = f(x)$ has been

- reflected in the *y*-axis
- translated *a* units horizontally (to the left if *a* is negative and to the right if *a* is positive)
- translated 1 unit vertically upwards

For $y = 1 + f(a - x)$ to have no *y*-intercepts, we require that its vertical asymptote lies on the *y*-axis.
The vertical translation has no effect on the horizontal movement of the graph.
The reflection in the *y*-axis means that the asymptote of $x = 2$ will be relocated to become $x = -2$.
A translation of 2 units to the right will then place the asymptote on the *y*-axis. So we require that $a = 2$. **(1 mark)**

Method 2

$$\begin{aligned} y &= 1 + f(a - x) \\ &= 1 + a - x + \frac{1}{a - x - 2} \end{aligned}$$

Subsitute $x = 0$ to get the *y*-intercept i.e.

$$y = 1 + a + \frac{1}{a - 2}$$

Since $1 + a + \frac{1}{a - 2}$ is undefined when $a = 2$, the graph can have no *y*-intercepts when $a = 2$. **(1 mark)**

Question 8 (7 marks)

a. Since the probability density function f is continuous at $x=a$, then

$$e^x - 1 = 6e^{-x} \quad \text{at } x=a$$ **(1 mark)**

therefore $e^a - 1 = 6e^{-a}$

$$e^a - 1 - 6e^{-a} = 0$$

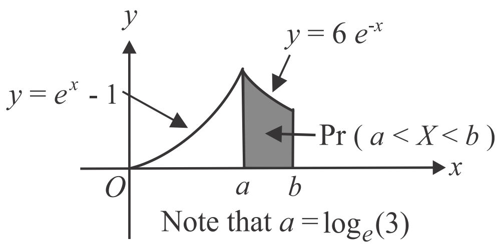
$$e^{2a} - e^a - 6 = 0 \quad \text{(multiply left and right hand sides by } e^a\text{)}$$

$$(e^a - 3)(e^a + 2) = 0$$

$$e^a = 3 \quad \text{or} \quad e^a = -2 \quad \text{but } e^a > 0 \text{ so reject this.}$$

So $a = \log_e(3)$. **(1 mark)**

b. **i.** $\Pr(a<X<b) = 1 - \Pr(0<X<a)$ since $\Pr(0<X<b) = 1$



$$\Pr(0<X<a) = \int_0^{\log_e(3)} (e^x - 1)\, dx$$

$$= \left[e^x - x\right]_0^{\log_e(3)}$$

$$= \left(e^{\log_e(3)} - \log_e(3)\right) - \left(e^0 - 0\right)$$

$$= 3 - \log_e(3) - 1$$

$$= 2 - \log_e(3)$$

(1 mark)

So $\Pr(a<X<b) = 1 - \left(2 - \log_e(3)\right)$

$$= \log_e(3) - 1$$ **(1 mark)**

ii. So $\displaystyle\int_{\log_e(3)}^{b} 6e^{-x}\,dx = \log_e(3) - 1$ (from part **b. i.**) **(1 mark)**

Now $\displaystyle\int_{\log_e(3)}^{b} 6e^{-x}\,dx = \left[-6e^{-x}\right]_{\log_e(3)}^{b}$

$$\begin{aligned}
&= \left(-6e^{-b}\right) - \left(-6e^{-\log_e(3)}\right) \quad \textbf{(1 mark)}\\
&= \left(-6e^{-b}\right) - \left(-6e^{\log_e(3)^{-1}}\right)\\
&= \left(-6e^{-b}\right) - \left(-6e^{\log_e\left(\frac{1}{3}\right)}\right)\\
&= \left(-6e^{-b}\right) - \left(-6\times\frac{1}{3}\right)\\
&= \left(-6e^{-b}\right) - (-2)\\
&= 2 - 6e^{-b}
\end{aligned}$$

So
$$\begin{aligned}
2 - 6e^{-b} &= \log_e(3) - 1\\
-6e^{-b} &= \log_e(3) - 3\\
6e^{-b} &= 3 - \log_e(3)\\
e^{-b} &= \frac{3 - \log_e(3)}{6}\\
-b &= \log_e\left(\frac{3 - \log_e(3)}{6}\right)\\
b &= -\log_e\left(\frac{3 - \log_e(3)}{6}\right)\\
b &= \log_e\left(\frac{6}{3 - \log_e(3)}\right)
\end{aligned}$$

(1 mark)

Question 9 (8 marks)

a. x-intercepts occur when $y = 0$

$$0 = \sqrt{3 - x}$$
$$x = 3$$

A is the point (3, 0).

y-intercepts occur when $x = 0$

$$y = \sqrt{3 - 0}$$
$$= \sqrt{3}$$

B is the point $\left(0, \sqrt{3}\right)$. **(1 mark)**

$$\text{gradient } = \frac{\sqrt{3} - 0}{0 - 3}$$
$$= -\frac{\sqrt{3}}{3}$$

Using the coordinates of point A, $y - 0 = -\frac{\sqrt{3}}{3}(x - 3)$

$$y = -\frac{\sqrt{3}}{3}x + \sqrt{3}$$ **(1 mark)**

b. Method 1

$$\text{area} = \int_0^3 \left(\sqrt{3 - x} - \left(-\frac{\sqrt{3}}{3}x + \sqrt{3}\right)\right)dx$$
$$= \int_0^3 \left(\left((3 - x)^{\frac{1}{2}}\right) + \frac{\sqrt{3}}{3}x - \sqrt{3}\right)dx$$
$$= \left[-\frac{2}{3}(3 - x)^{\frac{3}{2}} + \frac{\sqrt{3}}{6}x^2 - \sqrt{3}x\right]_0^3$$ **(1 mark)**

$$= \left(0 + \frac{3\sqrt{3}}{2} - 3\sqrt{3}\right) - \left(-\frac{2}{3} \times 3\sqrt{3} + 0 - 0\right) \quad \text{since } 3^{\frac{3}{2}} = \sqrt{27} = 3\sqrt{3}$$
$$= \frac{3\sqrt{3}}{2} - 3\sqrt{3} + 2\sqrt{3}$$
$$= \frac{3\sqrt{3}}{2} - \sqrt{3}$$
$$= \frac{\sqrt{3}}{2} \text{ square units}$$ **(1 mark)**

Method 2 – using ΔAOB

$$\text{area } = \int_0^3 \sqrt{3-x}\,dx - \frac{1}{2}\times 3\times\sqrt{3}$$

$$= \int_0^3 (3-x)^{\frac{1}{2}}\,dx - \frac{3\sqrt{3}}{2}$$

$$= \left[-\frac{2}{3}(3-x)^{\frac{3}{2}}\right]_0^3 - \frac{3\sqrt{3}}{2}$$ **(1 mark)**

$$= (0) - \left(-\frac{2}{3}\times 3^{\frac{3}{2}}\right) - \frac{3\sqrt{3}}{2}$$

$$= \frac{2}{3}\times 3\sqrt{3} - \frac{3\sqrt{3}}{2} \qquad \text{i.e. } 3^{\frac{3}{2}} = \sqrt{27} = 3\sqrt{3}$$

$$= 2\sqrt{3} - \frac{3\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{2}\text{ square units}$$ **(1 mark)**

c. $f(x) = \sqrt{3-x}$

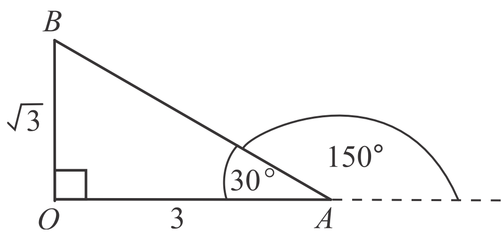
$$= (3-x)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(3-x)^{-\frac{1}{2}}\times -1 \qquad \text{(chain rule)}$$

$$= \frac{-1}{2\sqrt{3-x}}$$ **(1 mark)**

d. In ΔABO, $\tan(\angle BAO) = \frac{\sqrt{3}}{3}$

$$\angle BAO = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$



So the line AB makes an angle of 150° with the positive direction of the x-axis. When $\theta = 30^\circ$, the tangent to f at P makes an angle of 120° with the positive direction of the x-axis.

$$f'(x) = \tan(120^\circ)$$

$$= -\sqrt{3} \qquad \text{(tan is negative in the second quadrant)}$$

(1 mark)

Solve $\frac{-1}{2\sqrt{3-x}} = -\sqrt{3}$ for x (using our result to part **c.**)

$$\frac{1}{2\sqrt{3}} = \sqrt{3-x}$$

$$\frac{1}{4\times 3} = 3-x$$

$$x = 3 - \frac{1}{12}$$

$$x = \frac{35}{12}$$

(1 mark)

$$f\left(\frac{35}{12}\right) = \sqrt{3-\frac{35}{12}}$$

$$= \sqrt{\frac{1}{12}}$$

$$= \frac{1}{2\sqrt{3}}$$

P is the point $\left(\frac{35}{12}, \frac{1}{2\sqrt{3}}\right)$ or $\left(\frac{35}{12}, \frac{\sqrt{3}}{6}\right)$.

(1 mark)