VCE® Specialist Mathematics

Practice Written Examination 2

ADVICE FOR TEACHERS

Teachers should provide students with a copy of the VCAA formula sheet.

The VCAA formula sheet can be downloaded and printed from

https://www.vcaa.vic.edu.au/assessment/vce-assessment/past-examinations/Pages/Specialist-Mathematics.aspx

2024-MSP-VIC-U34-NA-EX2-QATS

Solution Pathway

Below are sample answers and solutions. Please consider the merit of alternative responses.

Specialist Mathematics Examination 2: Marking Scheme

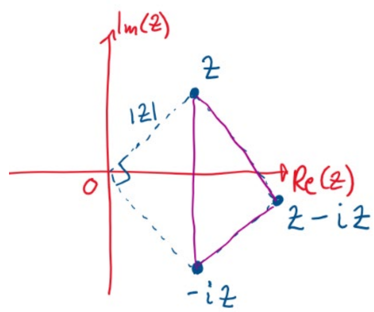
SECTION A: Multiple-Choice Questions – Answers

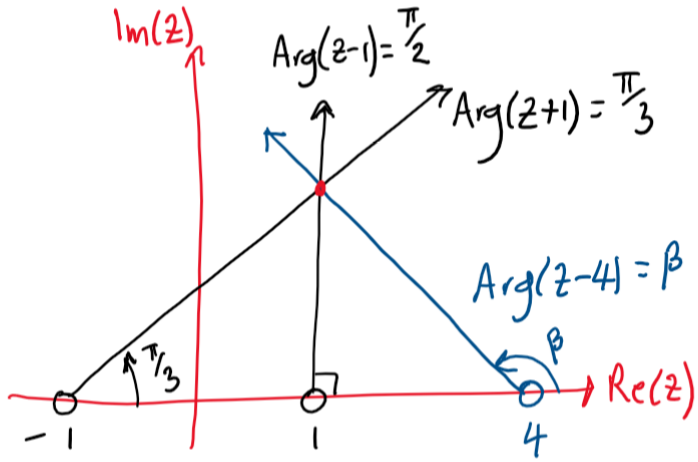
1.	C	**5.**	C	**9.**	A	**13.**	C	**17.**	C
2.	B	**6.**	D	**10.**	B	**14.**	D	**18.**	D
3.	A	**7.**	D	**11.**	C	**15.**	A	**19.**	C
4.	D	**8.**	B	**12.**	D	**16.**	B	**20.**	A

SECTION A: Multiple-Choice Questions – Solutions

Q 1	To prove that if $\underbrace{n^2-6n+5 \text{ is even}}_{P}\ \underbrace{\text{then}}_{\Rightarrow}\ \underbrace{n \text{ is odd}}_{Q}$, it is assumed for the sake of contradiction that $\underbrace{n^2-6n+5 \text{ is even}}_{P}\ \underbrace{\text{and}}_{\wedge}\ \underbrace{n \text{ is even}}_{\neg Q}$	**C**

Q 2		B
	• $x=6$ is a vertical asymptote therefore $x=6$ is a root of $ax^2+b\|x\|+c$: $36a+6b+c=0$ …. (1) • Let $g(x)=\frac{x^3-1}{ax^2+bx+c}$ and $f(x)=\frac{\|x\|^3-1}{ax^2+b\|x\|+c}=g(\|x\|)$ From a CAS: $g(x)=\frac{1}{a}x-\frac{b}{a^2}+\frac{\text{Linear}}{ax^2+bx+c}$ Therefore $y=g(x)$ has an oblique asymptote $y=\frac{1}{a}x-\frac{b}{a^2}$ • **Case 1:** $x\geq 0$: $f(x)=g(x)$ therefore $y=\frac{1}{a}x-\frac{b}{a^2}$ is a diagonal asymptote of $y=f(x)$. Compare $y=\frac{1}{a}x-\frac{b}{a^2}$ with $y=-2x+14$: $\frac{1}{a}=-2\Rightarrow a=-\frac{1}{2}$. $-\frac{b}{a^2}=14\Rightarrow b=-14a^2=-\frac{7}{2}$ Substitute $a=-\frac{1}{2}$ and $b=-\frac{7}{2}$ into (1): $c=39$. No corresponding option. • **Case 2:** $x<0$: $f(x)=g(-x)$ therefore $y=-\frac{1}{a}x-\frac{b}{a^2}$ is a diagonal asymptote of $y=f(x)$ Compare $y=-\frac{1}{a}x-\frac{b}{a^2}$ with $y=-2x+14$: $-\frac{1}{a}=-2\Rightarrow a=\frac{1}{2}$. $-\frac{b}{a^2}=14\Rightarrow b=-14a^2=-\frac{7}{2}$ Substitute $a=\frac{1}{2}$ and $b=-\frac{7}{2}$ into (1): $c=3$. **Option B**. **Discussion of Case 2:** It follows that $g(x)=\frac{x^3-1}{\frac{1}{2}x^2-\frac{7}{2}x+3}=\frac{2(x-1)\left(x^2+x+1\right)}{(x-1)(x-6)}=\frac{2\left(x^2+x+1\right)}{x-6}$, $x\neq 1$ Therefore $y=g(x)$ has a ‘hole’ at $x=1$.	

Q 3	Horizontal asymptotes: $y = a - b^2\left(\frac{\pi}{2}\right)$ and $y = a - b^2\left(-\frac{\pi}{2}\right) = a + b^2\left(\frac{\pi}{2}\right)$. It is therefore required that either: **Case 1:** $a - b^2\left(\frac{\pi}{2}\right) \geq 0$ and $a > 0$ therefore $b^2 \leq \frac{2a}{\pi}$ and $a > 0$ or **Case 2:** $a + b^2\left(\frac{\pi}{2}\right) \leq 0$ and $a < 0$ therefore $b^2 \leq -\frac{2a}{\pi}$ and $a < 0$ Both cases are equivalent to $b^2 \leq \left\lvert\frac{2a}{\pi}\right\rvert$	**A**
Q 4	The geometric interpretation of $-iz$ is the rotation of z by 90^0 in a clockwise direction. Therefore the area of the triangle is half the area of a square whose vertices are at 0, z, $-iz$, $z-iz$:  By inspection the length of a side of the square is $\lvert z \rvert$ therefore the area of the square is $\lvert z \rvert^2$ therefore the area of the triangle is $\frac{1}{2}\lvert z \rvert^2$.	**D**

Q 5	Let $z = x + iy$, $x, y \in R$. The rays $\text{Arg}(z+1) = \frac{\pi}{3}$, $\text{Arg}(z-1) = \frac{\pi}{2}$ and $\text{Arg}(z-4) = \beta$ intersect at the point where $x = 1$:  The cartesian equation of $\text{Arg}(z+1) = \frac{\pi}{3}$ is $y = \sqrt{3}x + \sqrt{3}$, $x > -1$ Substitute $x = 1$: $y = 2\sqrt{3}$ Therefore the rays $\text{Arg}(z+1) = \frac{\pi}{3}$, $\text{Arg}(z-1) = \frac{\pi}{2}$ and $\text{Arg}(z-4) = \beta$ intersect at $z = 1 + 2\sqrt{3}i$ Substitute $z = 1 + 2\sqrt{3}i$ into $\text{Arg}(z-4) = \beta$: $\beta = \text{Arg}\left(-3 + 2\sqrt{3}i\right) = \pi - \arctan\left(\frac{2\sqrt{3}}{3}\right)$	**C**

Q 6 **D**

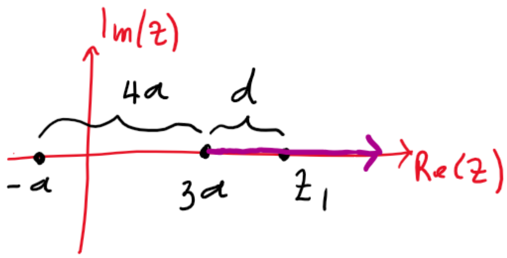
a is real therefore $3a$ and $-a$ lie on the real axis.

Geometric interpretation:

$|z+a|$ is the distance of z from $-a$, $|z-3a|$ is the distance of z from $3a$

The distance between $-a$ and $3a$ is $4a$

Let $z = z_1 \in R$ where $z_1 > 3a$. Let $|z_1 - 3a| = d \Rightarrow |z_1 + a| = 4a + d$:



Then $|z_1 + a| - |z_1 - 3a| = (4a + d) - d = 4a$

Therefore $|z+a| - |z-3a| = 4a$ defines a ray.

Discussion: By inspection, the equation of the ray is $y = 0$, $x \geq 3a$. The terminus of the ray has coordinates $(3a,\ 0)$ and is included as part of the ray.

Algebraic calculation:

Substitute $z = x + iy$, $x, y \in R$, into $|z+a| - |z-3a| = k$:

$$\sqrt{(x+a)^2 + y^2} - \sqrt{(x-3a)^2 + y^2} = k$$

$$\Rightarrow \sqrt{(x+a)^2 + y^2} = k + \sqrt{(x-3a)^2 + y^2}$$

$$\Rightarrow (x+a)^2 + y^2 = k^2 + (x-3a)^2 + y^2 + 2k\sqrt{(x-3a)^2 + y^2}$$

$$\Rightarrow 8ax - 8a^2 - k^2 = 2k\sqrt{(x-3a)^2 + y^2}$$

Note the implied restriction:

RHS ≥ 0 therefore LHS $= 8ax - 8a^2 - k^2 \geq 0 \Rightarrow x \geq \dfrac{8a^2 + k^2}{8a}$

$$\Rightarrow \left(8ax - 8a^2 - k^2\right)^2 = 4k^2\left((x-3a)^2 + y^2\right)$$

$$\Rightarrow (4a-k)(4a+k)(2x+k-2a)(2x-k-2a) = 4k^2y^2, \quad x \geq \frac{8a^2 + k^2}{8a}$$

If $k = 4a$ then a ray is obtained: $0 = 4k^2y^2 \Rightarrow y = 0$, $x \geq 3a$

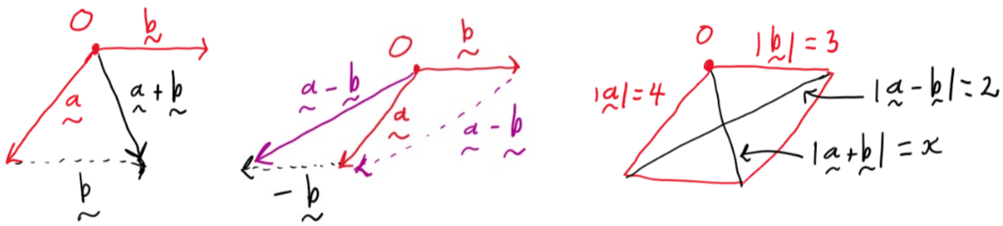
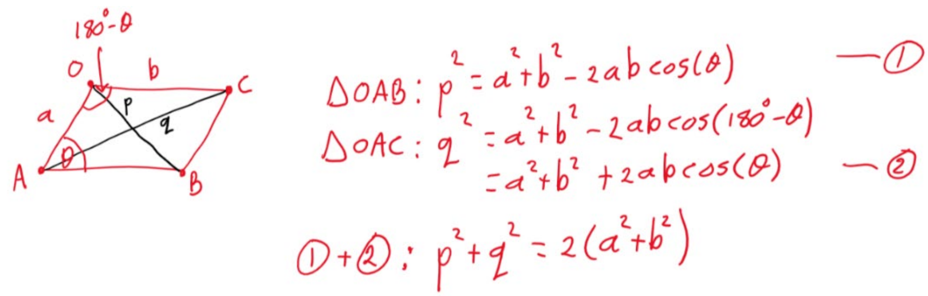
	If $k > 4a$: $x > 3a$ and there is no solution since LHS $= (-ve)(+ve)(+ve)(+ve) < 0$ and RHS > 0	
Q 7	The pseudocode applies Euler's method to find an approximate solution to the differential equation $\frac{dx}{dt} = \frac{t^2+1}{x^3+1} = f(t,x)$ given that $t_0 = a$ and $x_0 = 3$ The number of iterations is $n = 10$ therefore $t_{10} = b = 4$, the value of the step size is $\frac{b-a}{n} = \frac{4-a}{10}$ and $x_{10} = 3.504$ (output correct to three decimal places). Substitute the value of a in each option into the above data and execute Euler's Method on a CAS.	**D**

Q 8	$\frac{dm}{dt}=$ (rate at which salt enters the tank) – (rate at which salt leaves the tank). • Rate at which salt enters the tank = (rate of inflow) (concentration of salt in inflow) = (5 litres per minute) (6 kg per litre) = 30 kg per minute. • Rate at which salt leaves the tank = (rate of outflow) (concentration of salt in outflow) = (7 litres per minute) (concentration of salt in outflow). • After t minutes have elapsed: Concentration of salt in outflow $=\frac{m}{V}$ where V is the volume of solution in the tank after t minutes: $V=$ (initial volume) + (volume added after t minutes via inflow) – (volume removed after t minutes via outflow) = 90 litres + 5t litres – $7t$ litres $=90-2t$ litres. Therefore: concentration of salt in outflow $=\frac{m}{V}=\frac{m}{90-2t}$ kg per litre. Therefore rate at which salt leaves the tank = (7 litres per minute) (concentration of salt in outflow) $=(7)\left(\frac{m}{90-2t}\right)$ $=\frac{7m}{90-2t}$ kg per minute $\frac{dm}{dt}=$(rate at which salt enters the tank) – (rate at which salt leaves the tank) $=(30)-\left(\frac{7m}{90-2t}\right)=30-\frac{7m}{90-2t}$	**B**

	• Differential equation for the concentration *c* of salt in the tank at time t: $c = \frac{m}{V} = \frac{m}{90-2t} \quad \Rightarrow m = c(90-2t)$ $\Rightarrow \frac{dm}{dt} = \frac{d}{dt}\left(c(90-2t)\right) \quad = \underbrace{\frac{dc}{dt}(90-2t) + c(-2)}_{\text{Product Rule}} \quad = (90-2t)\frac{dc}{dt} - 2c$ Substitute $\frac{dm}{dt} = 30 - \frac{7m}{90-2t}$: $(90-2t)\frac{dc}{dt} - 2c = 30 - \frac{7c(90-2t)}{90-2t} \quad \Rightarrow (90-2t)\frac{dc}{dt} = 30 - 5c$	
Q 9	The value of $\frac{dy}{dx}$ at $t = 2$ is required. When $t > \sqrt{3}$: $x(t) = \frac{\lvert 3-t^2 \rvert}{t+1} = \frac{t^2-3}{t+1}$, $\quad y(t) = -\frac{4}{t+2}$. Use a CAS to get the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ when $t = 2$: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{9}{44}$ when $t = 2$	**A**

Q 10	Consider $J_n = \int \sec^n(x)\, dx$ and use integration by parts: $\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$ Let $u = \sec^{n-2}(x)$ $\Rightarrow \frac{du}{dx} = (n-2)\sec(x)^{n-3}\sin(x)\sec^2(x) = (n-2)\sec(x)^{n-1}\sin(x)$ Let $\frac{dv}{dx} = \sec^2(x) \quad \Rightarrow v = \tan(x)$ $J_n = \underbrace{\sec^{n-2}(x)}_{u}\underbrace{\tan(x)}_{v} - \int \underbrace{\tan(x)}_{v}\underbrace{(n-2)\sec^{n-1}(x)\sin(x)}_{\frac{du}{dx}}\, dx$ $= \sec^{n-2}(x)\tan(x) - (n-2)\int \sin^2(x)\sec^n(x)\, dx$ $= \sec^{n-2}(x)\tan(x) - (n-2)\int \left(1-\cos^2(x)\right)\sec^n(x)\, dx$ $= \sec^{n-2}(x)\tan(x) - (n-2)\left[\int \sec^n(x)\, dx - \int \sec^{n-2}(x)\, dx\right]$ $= \sec^{n-2}(x)\tan(x) - (n-2)\left[J_n - J_{n-2}\right]$ $= \sec^{n-2}(x)\tan(x) - (n-2)J_n + (n-2)J_{n-2}$ $\Rightarrow J_n = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}J_{n-2} \quad (n \neq 1)$ $\Rightarrow I_n = \frac{1}{n-1}\left[\sec^{n-2}(x)\tan(x)\right]_0^{\frac{\pi}{3}} + \frac{n-2}{n-1}I_{n-2} \quad = \frac{2^{n-2}}{n-1}\sqrt{3} + \frac{n-2}{n-1}I_{n-2}$	**B**

Q 11	The area of the surface generated by rotating the curve with equation $y = \tan^{-1}(x)$ about the line $x = 1$ from the points where $x = 2$ to $x = 4$ is equal to the surface generated by rotating the curve with equation $y = \tan^{-1}(x+1)$ about the *y*-axis from the points where $x = 1$ to $x = 3$. Area of the surface generated by rotating the curve $y = f(x)$ about the y-axis: $S = 2\pi \int_{y=a}^{y=b} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$ $y = \tan^{-1}(x+1) \Rightarrow \tan(y) = x+1 \Rightarrow x = \tan(y)-1$ $x = 1 \Rightarrow y = \tan^{-1}(2)$. $x = 3 \Rightarrow y = \tan^{-1}(4)$ $S = 2\pi \int_{\tan^{-1}(2)}^{\tan^{-1}(4)} \left(\tan(y)-1\right)\sqrt{1+\sec^4(y)}\, dy$	**C**

Q 12	The vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ form a parallelogram with pairs of opposite sides of length $\lvert\underset{\sim}{a}\rvert = 4$ and $\lvert\underset{\sim}{b}\rvert = 3$ and diagonals of length $\lvert\underset{\sim}{a} - \underset{\sim}{b}\rvert = 2$ and $\lvert\underset{\sim}{a} + \underset{\sim}{b}\rvert = x$:  For the parallelogram with pairs of opposite sides of length a and b and diagonals of length p and q: $p^2 + q^2 = 2\left(a^2 + b^2\right)$.  $p^2 + q^2 = 2\left(a^2 + b^2\right) \quad \Rightarrow 2^2 + x^2 = 2\left(4^2 + 3^2\right) \quad \Rightarrow x = \lvert\underset{\sim}{a} + \underset{\sim}{b}\rvert = \sqrt{46}$	**D**
Q 13	Let the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be θ: • $\underset{\sim}{a} \cdot \underset{\sim}{b} = 4 \Rightarrow \lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos(\theta) = 4$. …. (1) • $\lvert\underset{\sim}{a} \times \underset{\sim}{b}\rvert = 7 \Rightarrow \lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\sin(\theta) = 7$. …. (2) • $\frac{(2)}{(1)}$: $\tan(\theta) = \frac{7}{4} \Rightarrow \theta \approx 60.3^0$	**C**
Q 14	$v = \frac{dx}{dt} = \tan^{-1}(x) \qquad \Rightarrow \frac{dt}{dx} = \frac{1}{\tan^{-1}(x)}$ and $x = 2$ when $t = 3$. The integral solution must be used: $t = \int_2^x \frac{1}{\tan^{-1}(w)}\, dw + 3$.	**D**

	Substitute $x=5$: $t=\int_{2}^{5}\frac{1}{\tan^{-1}(w)}\,dw+3\approx 5.359$	
Q 15	The acceleration of the object is constant therefore the straight line motion formulae for constant acceleration can be used: • Motion after 10 seconds: Data: $u=6$ ms^{-1}, $v=-8$ ms^{-1}, $t=10$ seconds, $a=?$ $v=u+at \quad \Rightarrow -8=6+a(10) \quad \Rightarrow a=-1.4$ ms^{-2} • Motion after 12 seconds: Data: $u=6$ ms^{-1}, $t=12$ seconds, $a=-1.4$ ms^{-2}, $x=?$ $x=ut+\frac{1}{2}at^2 \quad =6(12)+\frac{1}{2}(-1.4)(12)^2 \quad =-28.8$ m Therefore the distance is 28.8 metres.	**A**

Q 16

- Time at which the aircraft drops the package: $540 = 600 - 3t \Rightarrow t = 20$ seconds.
- Initial conditions for the motion of the package:

$\underset{\sim}{\text{r}}_P(0) = \underset{\sim}{\text{r}}(20) = 1000\,\underset{\sim}{\text{i}} + 540\,\underset{\sim}{\text{j}}$, $\quad \dot{\underset{\sim}{\text{r}}}_P(0) = \dot{\underset{\sim}{\text{r}}}(20) = 50\,\underset{\sim}{\text{i}} - 3\,\underset{\sim}{\text{j}}$

Method 1:

Solve $\ddot{\underset{\sim}{\text{r}}}_P(t) = -9.8\,\underset{\sim}{\text{j}}$ (since the air resistance acting on the package is negligible) subject to the above initial conditions (either using a CAS or 'by hand'):

$\underset{\sim}{\text{r}}_P(t) = (1000 + 50t)\,\underset{\sim}{\text{i}} + \left(540 - 3t - 4.9t^2\right)\underset{\sim}{\text{j}}$

Therefore $x_P(t) = 1000 + 50t$ and $y_P(t) = 540 - 3t - 4.9t^2$

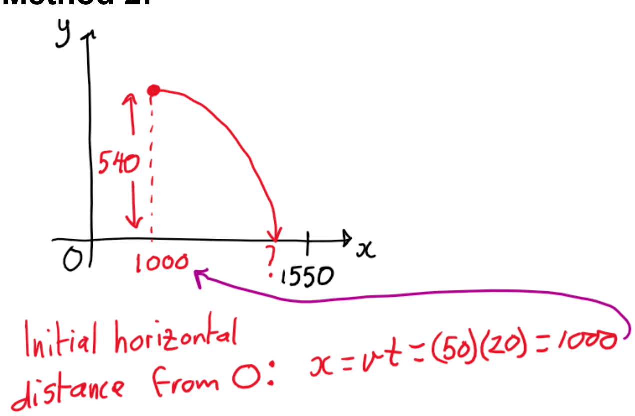
- Time at which the package lands:

$y_P(t) = 540 - 3t - 4.9t^2 = 0$ (and $t > 0$) $\quad \Rightarrow t = 10.1962$

- Horizontal distance of package from O:

$x_P(10.1962) = 1000 + 50(10.1962) = 1509.81\,\text{m}$

Method 2:



- Acceleration of the package is constant therefore the straight line motion formulae for constant acceleration can be used:

Horizontal motion ($\rightarrow +ve$):

$u = 50\ \text{ms}^{-1}$, $\quad a = 0\ \text{ms}^{-2}$

$y = 540\,\text{m}$

$t = ?\ \ 10.1962$, $\quad x = ?$

$x = ut + \frac{1}{2}at^2 = (50)(10.1962) = 509.81$

Vertical motion ($\downarrow +ve$):

$u = 3\ \text{ms}^{-1}$, $\quad a = 9.8\ \text{ms}^{-2}$,

$t = ?$

$y = ut + \frac{1}{2}at^2 \Rightarrow 540 = 3t + 4.9t^2$

$\Rightarrow t \approx 10.1962$

B

	• Therefore horizontal distance of package from O $=1000+509.81=1509.81$ • Therefore distance short of the drop position $=1550-1509.81=40.19$ m	
Q 17	• Determine the value(s) of k for which the planes $2x+(k-4)y+(3-k)z=1$, $2y+(k-3)z=2$, $x-2y+z=1$ do not intersect at a unique point: $\det\begin{vmatrix}2 & k-4 & 3-k\\0 & 2 & k-3\\1 & -2 & 1\end{vmatrix}=0 \quad \Rightarrow k=-1,2$ • Test each solution. $\underline{k=-1}$: $2x-5y+4z=1$, $y-2z=1$, $x-2y+z=1$ Solve simultaneously (use a CAS): Infinite number of solutions which corresponds to a line. $\underline{k=2}$: $2x-2y+z=1$, $2y-z=2$, $x-2y+z=1$ Solve simultaneously (use a CAS): No solution.	**C**

Q 18	• $\frac{x-1}{3}=\frac{3-y}{\alpha}=\frac{z+1}{4} \quad \Rightarrow x=3t+1,\ y=-\alpha t+3,\ z=4t-1,\ t\in R$ Therefore a vector in the direction of the line is $3\underset{\sim}{i}-\alpha\underset{\sim}{j}+4\underset{\sim}{k}$ • $\frac{x+1}{\beta}=\frac{y-2}{3}=\frac{z-1}{5} \quad \Rightarrow x=\beta s-1,\ y=3s+2,\ z=5s+1,\ s\in R$ Therefore a vector in the direction of the line is $\beta\underset{\sim}{i}+3\underset{\sim}{j}+5\underset{\sim}{k}$ • Therefore a vector normal to each line is (use a CAS) is $\left(3\underset{\sim}{i}-\alpha\underset{\sim}{j}+4\underset{\sim}{k}\right)\times\left(\beta\underset{\sim}{i}+3\underset{\sim}{j}+5\underset{\sim}{k}\right)=(-12-5\alpha)\underset{\sim}{i}+(4\beta-15)\underset{\sim}{j}+(9+\alpha\beta)\underset{\sim}{k}$ • Therefore $3\underset{\sim}{i}-7\underset{\sim}{j}+\gamma\underset{\sim}{k}=(-12-5\alpha)\underset{\sim}{i}+(4\beta-15)\underset{\sim}{j}+(9+\alpha\beta)\underset{\sim}{k}$ Equate $\underset{\sim}{i}$-components: $3=-12-5\alpha\Rightarrow\alpha=-3$ Equate $\underset{\sim}{j}$-components: $-7=4\beta-15\Rightarrow\beta=2$ Equate $\underset{\sim}{k}$-components: $\gamma=9+(-3)(2)=3$	**D**
Q 19	• $\text{sd}\left(\overline{X}\right)=\frac{\text{sd}(X)}{\sqrt{n}}$. • $\text{Var}(X)=E\left(X^2\right)-\left(E(X)\right)^2=\int_0^1 x^2(2x)\,dx-\left(\int_0^1 x(2x)\,dx\right)^2=\frac{1}{18}$ $\Rightarrow \text{sd}(X)=\frac{1}{\sqrt{18}}$ • $\text{sd}\left(\overline{X}\right)=\frac{\text{sd}(X)}{\sqrt{n}}=\frac{\frac{1}{\sqrt{18}}}{\sqrt{80}}=\frac{\sqrt{10}}{120}$	**C**

Q 20	**Method 1:** Use a CAS to check the confidence interval arising from the value of n in each option: $n = 92$. **Method 2:** $14572 = \bar{x} + k\frac{\sigma}{\sqrt{n}}$ .... (1) $14428 = \bar{x} - k\frac{\sigma}{\sqrt{n}}$ .... (2) $(1) + (2) \Rightarrow \bar{x} = 14500$ For a 90% confidence interval: $\Pr(-k < Z < k) = 0.95 \Rightarrow k \approx 1.64485$ (use a CAS). Substitute $\bar{x} = 14500$, $k = 1.64485$ and $\sigma = 420$ into either equation (1) or equation (2) and solve for n (use a CAS and round to the nearest integer): $n = 92$.	**A**

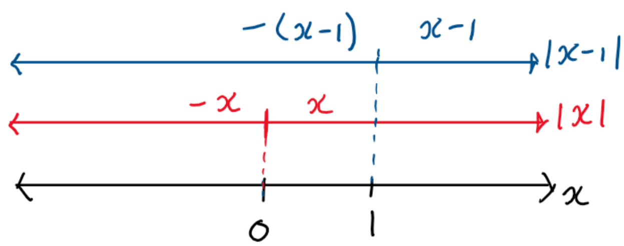
SECTION B: Solutions

Question 1

Part a.:

Note: These are only potential asymptotes because the asymptotic behaviour of f depends on the values of a and b. Some of these asymptotes do not exist for particular values of a and b (for example, *f* has no asymptotes when $a = 0$ and $b = -1$).

- Potential vertical asymptote: $x = 1$
- Asymptotic behaviour as $x \to \pm\infty$:



To consider $x \to +\infty$ it is therefore necessary to get the rule for $y = f(x)$ when $x \geq 1$:

$$y = f(x) = \frac{x^2 - ax + b}{x - 1} = \underbrace{x + 1 - a + \frac{b - a + 1}{x - 1}}_{\text{Use a CAS or 'by hand'}} \qquad x \to +\infty: \; y \sim x + 1 - a$$

To consider $x \to -\infty$ it is therefore necessary to get the rule for $y = f(x)$ when $x < 0$:

$$y = f(x) = \frac{x^2 - a(-x) + b}{-(x-1)} = \frac{x^2 + ax + b}{-x + 1} = \underbrace{-x - 1 - a - \frac{a + b + 1}{x - 1}}_{\text{Use a CAS or 'by hand'}} \qquad x \to -\infty: \; y \sim -x - 1 - a$$

Three correct potential asymptotes: $x = 1$, $y = x + 1 - a$, $y = -x - 1 - a$ Do **not** accept $y = \|x+1\| - a$ in place of $y = x + 1 - a$ and $y = -x - 1 - a$ (because $y = x + 1 - a$ is sometimes a diagonal asymptote when $y = -x - 1 - a$ is not, and vice versa).	**2 marks**
Two correct potential asymptotes.	**1 mark**
One or no correct potential asymptotes.	**0 marks**

Part b.:

$$y = b \Rightarrow \frac{x^2 - a|x| + b}{|x-1|} = b \Rightarrow x = 0$$

For f to be 'smooth' at $x = 0$ it is necessary and sufficient that $f'(x)$ is continuous at $x = 0$

It is therefore necessary and sufficient that:

1. $\lim_{x \to 0} f'(x)$ exists.   2. $\lim_{x \to 0} f'(x) = f'(0)$

1. Existence of $\lim_{x \to 0} f'(x)$

From a CAS:

$\lim_{x \to 0^+} f'(x) = b - a$ $\lim_{x \to 0^-} f'(x) = a + b$

$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^-} f'(x) \quad \Rightarrow b - a = a + b \quad \Rightarrow a = 0$ and $b \in R$

$\lim_{x \to 0^+} f'(x) = b - a$ $\lim_{x \to 0^-} f'(x) = a + b$ $a = 0$ and $b \in R$	**1 mark**

2. $a = 0$ and $b \in R \Rightarrow \lim_{x \to 0} f'(x) = b$. It must now be checked that $f'(0) = b$

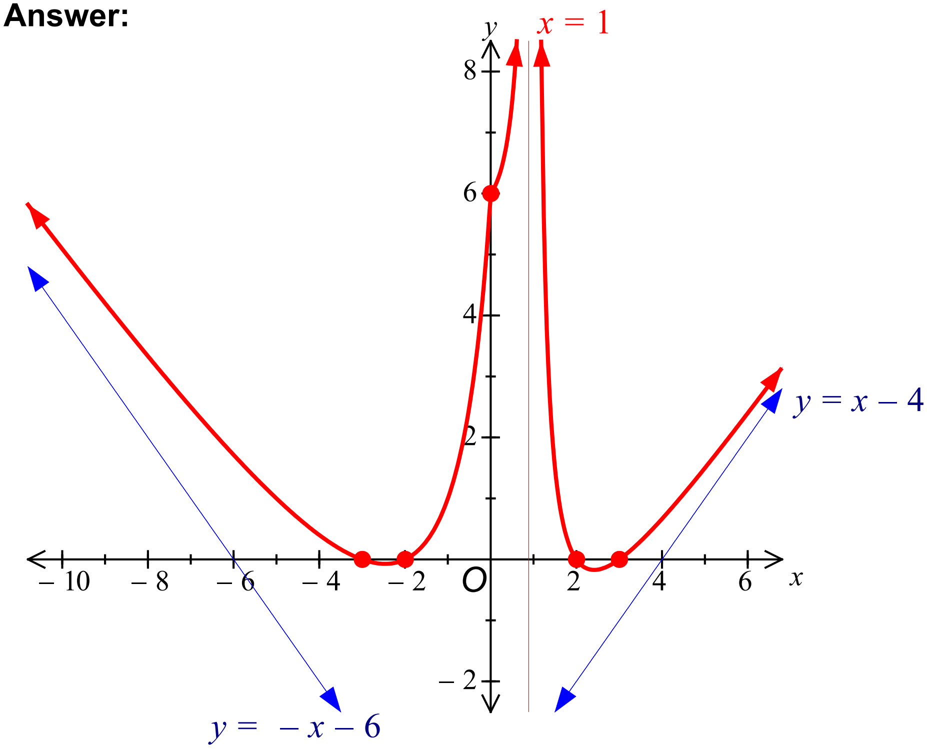
$a = 0$ and $x \to 0 \Rightarrow x < 1$ therefore $f(x) = \frac{x^2 + b}{-(x-1)} = \frac{x^2 + b}{-x+1}$

From a CAS: $f'(0) = b$

$\lim_{x \to 0} f'(x) = b$ $f(x) = \frac{x^2 + b}{-x+1}$ $f'(0) = b$ Therefore not 'smooth' at the point where $y = b$ unless $a = 0$	**1 mark**

Part c.:

Answer:



Shape: There must be a 'corner' at the y-intercept. The turning point in the interval $(2,\ 3)$ must be lower than the turning point in the interval $(-3,\ -2)$.	**1 mark**
Asymptotes: $x=1$, $y=x-4$, $y=-x-6$.	**1 mark**
Axis intercepts: $(2,\ 0)$, $(3,\ 0)$, $(-2,\ 0)$, $(-3,\ 0)$, $(0,\ 6)$.	**1 mark**

Calculations:

- Asymptotes are found from **Part a.**

Vertical asymptote: $x=1$

Diagonal asymptotes: Substitute into $a=5$ into $y=x+1-a$ and $y=-x-1-a$

- Axis intercepts:

y-intercept: Substitute $x=0$ x-intercepts: Use a CAS to solve $\dfrac{x^2-5|x|+6}{|x-1|}=0$

- Shape: From **Part b.** the graph is not smooth at $x=0$ (that is, the y-intercept) because $a \neq 0$. Therefore there is a 'corner' at $x=0$

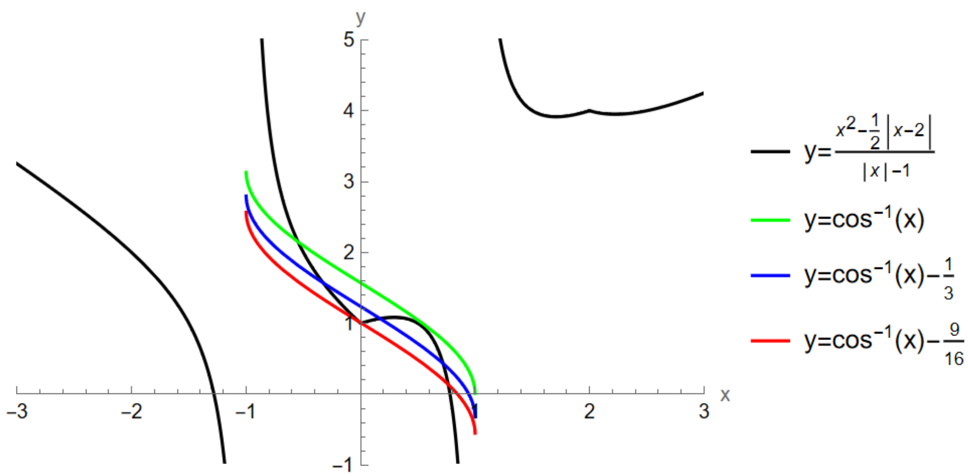
Part d.:

- $\dfrac{x^2 - \frac{1}{2}|x-2|}{|x|-1} - \arccos(x) - k = 0 \quad \Rightarrow \dfrac{x^2 - \frac{1}{2}|x-2|}{|x|-1} = \arccos(x) + k$

- Consider the graphs of $y = f(x) = \dfrac{x^2 - \frac{1}{2}|x-2|}{|x|-1}$ and $y = h(x) = \arccos(x) + k$

It is required to find the values of k for which these graphs have three intersection points

The value of k controls the vertical translation of $y = \arccos(x-1) + k$:



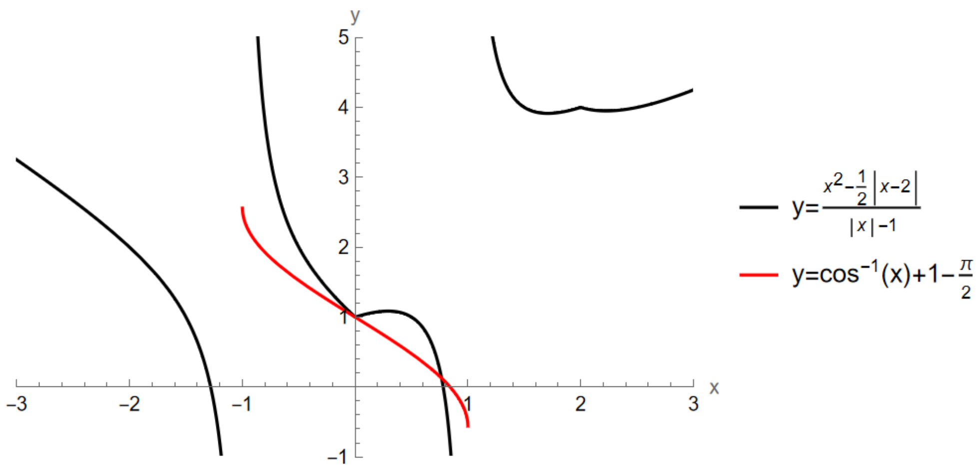
=

- Minimum value of k:

By inspection the value of k needs to be greater than the value of k such that point of inflection of $h(x) = \arccos(x) + k$ coincides with the 'corner' of $f(x) = \dfrac{x^2 - \frac{1}{2}|x-2|}{|x|-1}$ at (1, 0).

It is therefore required that the y-coordinate of the point of inflection of $y = \arccos(x) + k$ is equal to 1:

$$h(0) = 1: \ \frac{\pi}{2} + k = 1 \qquad \Rightarrow k = 1 - \frac{\pi}{2}$$



- Maximum value of k:

By inspection there is some value $k = \alpha$ such that $f(x) = \dfrac{x^2 - \frac{1}{2}|x-2|}{|x|-1}$ and $h(x) = \arccos(x) + k$ have a common tangent in the interval $0 < x < 1$

There are therefore three intersection points when $1 - \dfrac{\pi}{2} < k \le \alpha$

$f(x) = \dfrac{2x^2 + x - 2}{2(x-1)}$ for $0 < x < 1$

Let $x = \beta$ at the point of common tangency. It is required that

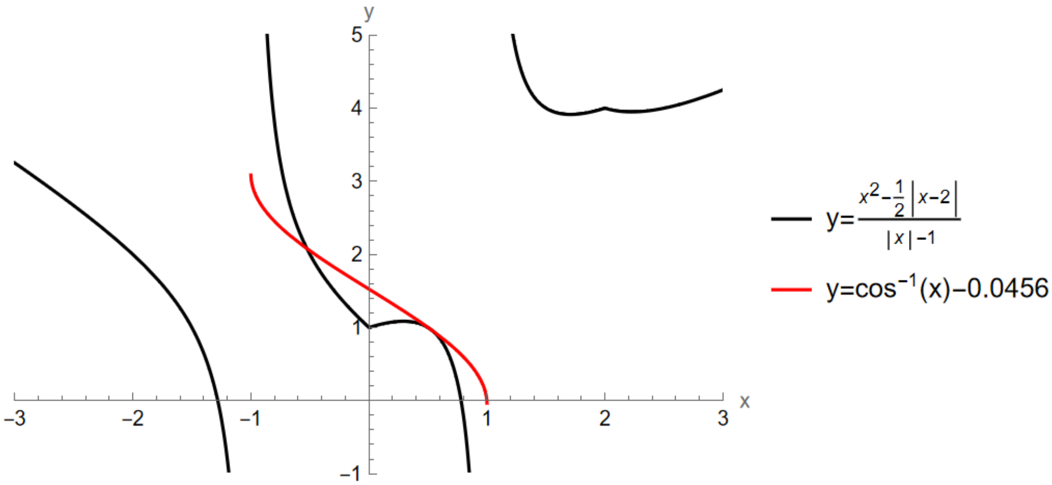
$f(\beta) = h(\beta) \quad \ldots . (1)$

$f'(\beta) = h'(\beta) \quad \ldots . (2)$

and $0 < \beta < 1$

Use a CAS to solve equations (1) and (2) simultaneously for α : $\alpha \approx -0.045612$

Note: $\beta \approx 0.520072$



$1-\frac{\pi}{2} < k \le -0.0456$. **Note:** The correct maximum value of k, rounded to four decimal places, is $-0.045\underline{7}$ and should be accepted (rounding UP is required). This is because $y = \arccos(x) - 0.045\underline{6}$ is above $y = \frac{x^2 - \frac{1}{2}\|x-2\|}{\|x\|-1}$ on the interval $(0,\ 1)$	**2 marks:** 1 mark for the minimum value of k. 1 mark for the maximum value of k.

Question 2

Part a.:

$|z-1|=1$ is a circle with radius $r=1$ and centre at $z=1$:

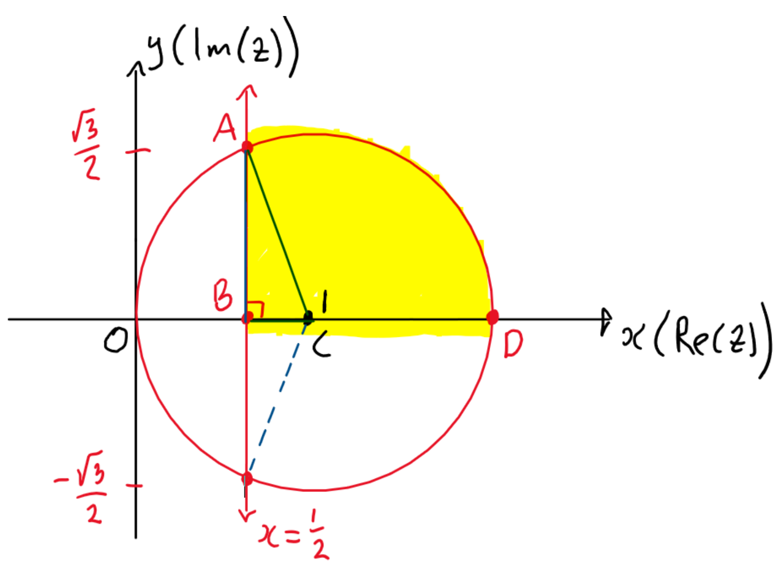
Answer: $(x-1)^2+y^2=1$	**1 mark**

Part b.:

- $|z|=|z-1|$ is the perpendicular bisector of the line segment joining $z=0$ and $z=1$: $x=\frac{1}{2}$
- Solve $(x-1)^2+y^2=1$ and $x=\frac{1}{2}$ simultaneously to get coordinates of intersection points:

Answer: $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$	**1 mark**

- Required area $=2\left(\text{Area of sector } ACD+\text{Area of triangle } ABC\right)$:



- Area of sector ACD:

$$\cos(\angle ACB) = \frac{BC}{AC} = \frac{1}{2} \quad \Rightarrow \angle ACB = \frac{\pi}{3} \quad \Rightarrow \angle DCB = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Area of sector $ACD = \frac{1}{3}(\underbrace{\text{Area of circle}}_{r=1}) = \frac{\pi}{3}$

Area of sector $ACD = \frac{\pi}{3}$	**1 mark**

- Area of triangle $ABC = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{8}$

- Required area $= 2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right)$

Answer: $\frac{2\pi}{3} + \frac{\sqrt{3}}{4}$ Accept $\frac{8\pi + 3\sqrt{3}}{12}$	**1 mark**

Part c.:

- $z=0$ is the exceptional element of $S=\{z: \ |z-1|=1, \ z\in C\}$ that is not an element of

$$\left\{z: \ a\text{Arg}\left(1-\frac{b}{z}\right)=\text{Arg}\left(z^2\right), \ z\in C\right\}$$

- All elements $S\setminus\{0\}$ are elements of $\left\{z: \ a\text{Arg}\left(1-\frac{b}{z}\right)=\text{Arg}\left(z^2\right), \ z\in C\right\}$

Therefore substitute two convenient elements of $S\setminus\{0\}$ (use the answer to **Part a.**). For example:

$\underline{z=1+i}$: $a\text{Arg}\left(1-\frac{b}{1+i}\right)=\text{Arg}\left((1+i)^2\right)$ .... (1)

$\underline{z=\frac{1}{2}+\frac{\sqrt{3}}{2}i}$: $a\text{Arg}\left(1-\frac{b}{\frac{1}{2}+\frac{\sqrt{3}}{2}i}\right)=\text{Arg}\left(\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2\right)$ .... (2)

Use a CAS to solve equations (1) and (2) simultaneously under the restriction $a,b\in R^+$:

$a=1$ and $b=1$.

Note: The very convenient value $z=2$ does not give an equation that allows the unique solution for a and b.

Clear and valid method	**1 mark**
Answer: $a=1$ and $b=1$	**1 mark**

Part d.:

$-\frac{\pi}{2} < \text{Arg}(z) < \frac{\pi}{2}$ for $z \in S = \{z: \ |z| = |z-1|, \ z \in C\}$

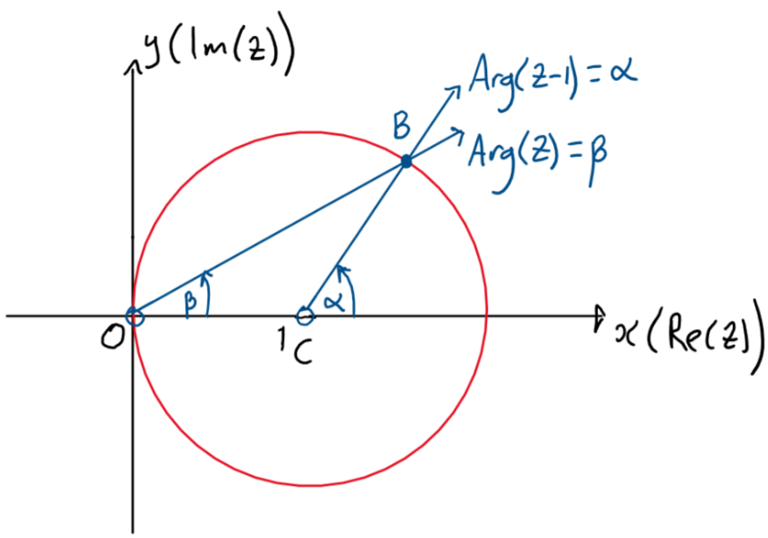
Note:

1. $z \to 0$ as $\text{Arg}(z) \to \pm\frac{\pi}{2}$ and $\text{Arg}(0)$ is not defined.
2. $\text{Arg}(z^2) = 2\text{Arg}(z)$ since $-\frac{\pi}{2} < \text{Arg}(z) < \frac{\pi}{2}$
3. $z = 0$ is the exceptional element of *S* that is not an element of $\{z: \ \text{Arg}(z-1) = \text{Arg}(z^2), \ z \in C\}$

(because $\text{Arg}(0^2)$ is not defined).

Case 1: $z \in S \setminus \{0\}$ and $0 \le \text{Arg}(z) < \frac{\pi}{2}$ (upper half of the circle).

Let $\text{Arg}(z-1) = \alpha$ and $\text{Arg}(z) = \beta$:



Proving $\text{Arg}(z-1) = \text{Arg}(z^2)$ is equivalent to proving $\alpha = 2\beta$:

$OC = CB$ $(= 1 =$ radius of circle)

therefore ΔOCB is isosceles

therefore $\angle BOC = \beta = \angle OBC$

therefore $\angle OCB = \pi - 2\beta$. .... (1)

But $\angle OCB = \pi - \alpha$. .... (2)

From equations (1) and (2): $\pi - 2\beta = \pi - \alpha \quad \Rightarrow 2\beta = \alpha$ ■

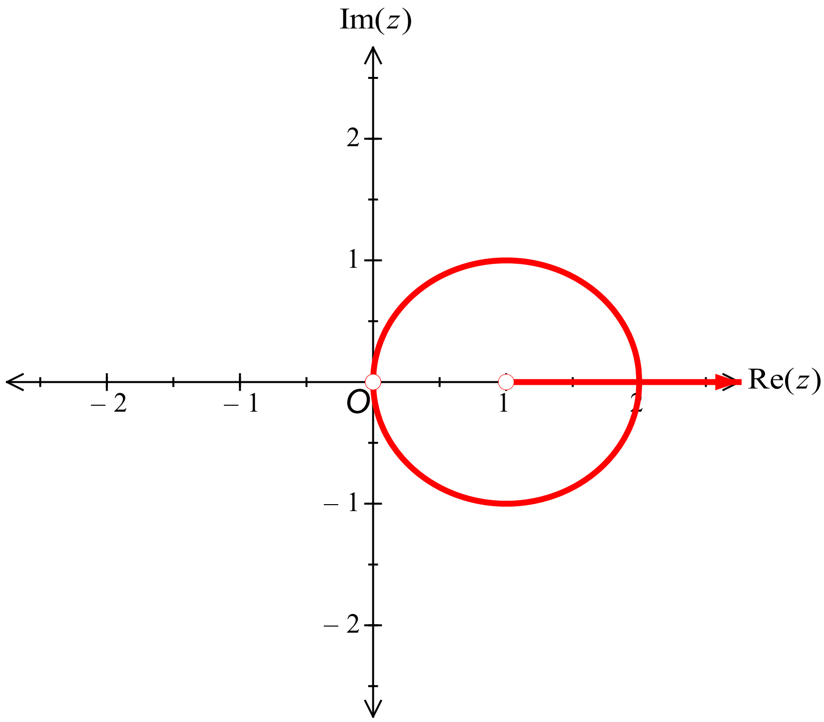
Case 2: $z \in S \setminus \{0\}$ and $-\frac{\pi}{2} < \text{Arg}(z) \le 0$ (lower half of the circle).

The proof is identical to **Case 1** by symmetry.

$\text{Arg}\left(z^2\right) = 2\text{Arg}(z)$ since $-\frac{\pi}{2} < \text{Arg}(z) < \frac{\pi}{2}$ $z = 0$ is the exceptional element of S that is not an element of $\left\{z:\ \text{Arg}(z-1) = \text{Arg}\left(z^2\right),\ z \in C\right\}$	**1 mark**
Appropriate labelled diagram. Geometric proof.	**1 mark**

Part e.:

Answer:



Shape: Circle $\|z-1\|=1$ with 'hole' at $z=0$	**1 mark**
Shape: Ray with terminus at $z=1$	**1 mark**
'Hole' at terminus of ray.	**1 mark**

Calculations:

Shape:

- From **part d.** it is known that the circle $|z-1|=1$, $z \neq 0$, is part of the solution.
- From **part d.** it is seen that $\text{Arg}(z-1)$ and $\text{Arg}\left(z^2\right)$ are equal when z is real and $z>1$ $(\alpha=\beta=0)$

Alternatively, $\text{Arg}(z-1)=\text{Arg}\left(z^2\right)$ when $z>1$ can be seen by inspection. If z is real:

$z^2>0$ therefore $\text{Arg}\left(z^2\right)=0$

$z>1 \Rightarrow z-1>0$ therefore $\text{Arg}(z-1)=0$ therefore $\text{Arg}(z-1)=\text{Arg}\left(z^2\right)$

$z<1 \Rightarrow z-1<0$ therefore $\text{Arg}(z-1)=\pi$ therefore $\text{Arg}(z-1)\neq\text{Arg}\left(z^2\right)$

$\text{Arg}(z-1)$ is not defined for $z=1$ therefore $z=1$ is not a solution to $\text{Arg}(z-1)=\text{Arg}\left(z^2\right)$

Part f.:

Let $\text{Arg}(z-1)=\alpha$ and $\text{Arg}(z)=\beta$ where $\alpha=2\beta$ (from **Part d.**).

- $\text{Arg}\left(z^2\right)=2\text{Arg}(z)$

  since $z\in S$ and $-\frac{\pi}{2}<\text{Arg}(z)<\frac{\pi}{2}$ for *S*

  $=2\beta$

- $\text{Arg}\left(z^2-z\right)=\text{Arg}\left(z(z-1)\right)=\text{Arg}(z)+\text{Arg}(z-1)=\beta+\alpha=\beta+2\beta=3\beta$

provided

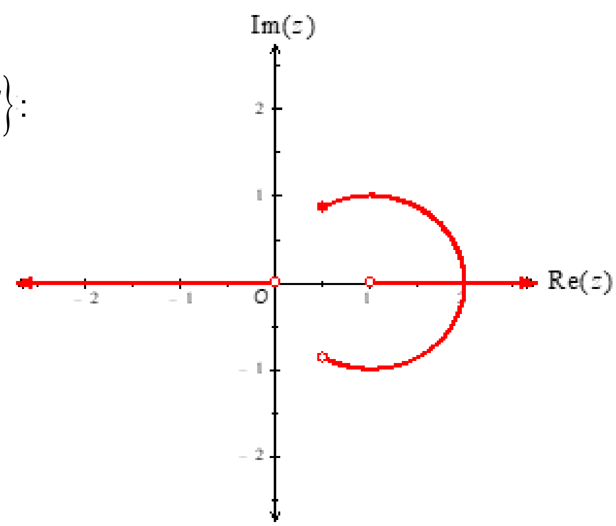
$-\pi<3\beta\le\pi \qquad \Rightarrow -\frac{\pi}{3}<\beta\le\frac{\pi}{3}$

Then $3\text{Arg}\left(z^2\right)=2\text{Arg}\left(z^2-z\right) \qquad \Rightarrow 3(2\beta)=2(3\beta)$ ✓

Recognition that $\text{Arg}\left(z^2-z\right)=\text{Arg}(z)+\text{Arg}(z-1)$ provided $-\pi<\text{Arg}(z)+\text{Arg}(z-1)\le\pi$	1 mark
Answer: $-\frac{\pi}{3}<\text{Arg}(z)\le\frac{\pi}{3}$	1 mark

Discussion:

Sketch of $\left\{z:\ 3\text{Arg}\left(z^2\right)=2\text{Arg}\left(z^2-z\right),\ z\in C\right\}$:



Calculations:

- It is known that the circle $|z-1|=1$ is part of the solution for $-\frac{\pi}{3}<\text{Arg}(z)\le\frac{\pi}{3}$
- It is seen by inspection that $3\text{Arg}\left(z^2\right)$ and $2\text{Arg}\left(z^2-z\right)$ are equal when *z* is real and $z^2>0$ and $z^2-z>0$:

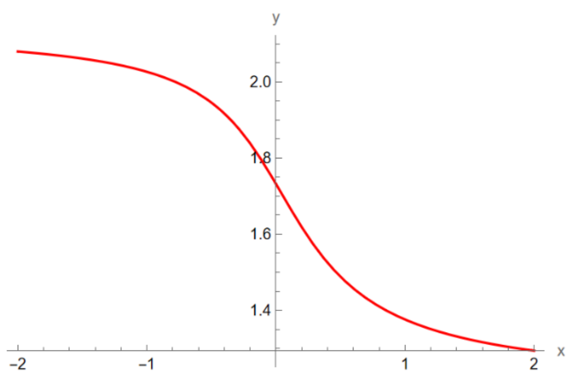
$z<0$ or $z>1$

Note: If z is real:

$z^2 > 0$ therefore $\text{Arg}\left(z^2\right) = 0$

- $z > 1 \Rightarrow z^2 - z > 0$ therefore $\text{Arg}\left(z^2 - z\right) = 0$ therefore $3\text{Arg}\left(z^2\right) = 2\text{Arg}\left(z^2 - z\right)$
- $0 < z < 1 \Rightarrow z^2 - z < 0$ therefore $\text{Arg}\left(z^2 - z\right) = \pi$ therefore $3\text{Arg}\left(z^2\right) \neq 2\text{Arg}\left(z^2 - z\right)$
- $z < -1 \Rightarrow z^2 - z > 0$ therefore $\text{Arg}\left(z^2 - z\right) = 0$ therefore $3\text{Arg}\left(z^2\right) = 2\text{Arg}\left(z^2 - z\right)$

Question 3



Part a.:

$V = \pi \int_0^1 \left(\underbrace{3 - \tan^{-1}(2x)}_{y}\right)^2 dx$	**1 mark**

Answer: 7.2106 cubic units	**1 mark**

Part b.:

Integral terminals:

• $x=-1 \quad \Rightarrow y=\sqrt{3-\tan^{-1}(-2)}$. Accept $y=\sqrt{3+\tan^{-1}(2)}$ • $x=0 \quad \Rightarrow y=\sqrt{3}$	**1 mark**

$S=2\pi\int_{\sqrt{3}}^{\sqrt{3-\tan^{-1}(-2)}} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ or $S=2\pi\int_{\sqrt{3}}^{\sqrt{3+\tan^{-1}(2)}} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ Accept $S=-2\pi\int_{\sqrt{3-\tan^{-1}(-2)}}^{\sqrt{3}} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$, $S=2\pi\left\|\int_{\sqrt{3-\tan^{-1}(-2)}}^{\sqrt{3}} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy\right\|$ *etc*	**1 mark** Order of terminals must be correct.

- Substitute $x=\frac{1}{2}\tan\left(3-y^2\right)$:

Answer: 3.2253 square units.	**1 mark**

Part c.:

$L=\pi\int_{-1}^{1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$	**1 mark**

Answer: 2.1233 units.	**1 mark**

Part d.:

Use $S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$:

- $S_f = 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\ dx$
- $S_g = 2\pi \int_a^b g(x)\sqrt{1+\left(g'(x)\right)^2}\ dx$

$S_g = 2\pi \int_a^b \left(f(x)+k\right)\sqrt{1+\left(f'(x)\right)^2}\ dx$	**1 mark**

$$= 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\ dx + 2\pi \int_a^b k\sqrt{1+\left(f'(x)\right)^2}\ dx$$

$$= 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\ dx + 2\pi k \underbrace{\int_a^b \sqrt{1+\left(f'(x)\right)^2}\ dx}_{L}$$

Answer: $S_g = S_f + 2\pi kL$	**1 mark**

Question 4

Part a.:

Require $\frac{dN}{dt} > 0$ at $t = 0$, that is, when $N = 150$: $2(150)\left(1-\frac{150}{600}\right) - n > 0$

Answer: $0 < n < 225$	**1 mark**

Part b.:

$$\frac{dN}{dt} = 2N\left(1 - \frac{N}{600}\right) - n \qquad \Rightarrow \frac{dt}{dN} = \frac{1}{2N\left(1 - \frac{N}{600}\right) - n} = \frac{1}{\frac{-N^2}{300} + 2N - n}$$

Require $\frac{-N^2}{300} + 2N - n$ to be a perfect square.	1 mark

Therefore $\Delta = (2)^2 - 4\left(\frac{-1}{300}\right)(-n) = 0$.

Answer: $n = 300$	1 mark

Check (use a CAS):

Solve $\frac{dN}{dt} = 2N\left(1 - \frac{N}{600}\right) - 300$ with initial condition $N(0) = 150$:

$$N = \frac{300(t-1)}{t-2} = 300 + \frac{300}{t-2}.$$

Observation: $N = 0$ when $t = 1$.

Part c.:

Use a CAS to solve $\frac{dN}{dt} = 2N\left(1 - \frac{N}{600}\right) - n$ with initial condition $N(0) = 150$:

$$N = 300 - 10\sqrt{3}\sqrt{n-300}\tan\left(\frac{\sqrt{n-300}}{10\sqrt{3}}t + \tan^{-1}\left(\frac{5\sqrt{3}}{\sqrt{n-300}}\right)\right)$$

Note: Different CAS may give different but equivalent forms.

Substitute $t = 4$ and solve $N = 0$ for *n* (round to the nearest integer):

$300 - 10\sqrt{3}\sqrt{n-300}\tan\left(\frac{4\sqrt{n-300}}{10\sqrt{3}} + \tan^{-1}\left(\frac{5\sqrt{3}}{\sqrt{n-300}}\right)\right) = 0$	**1 mark**

Answer: $n = 227$	**1 mark**

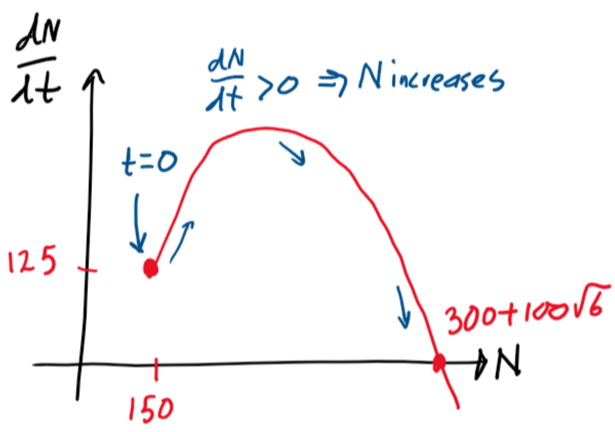
Check (use a CAS):

- Solve $\frac{dN}{dt} = 2N\left(1 - \frac{N}{600}\right) - 227$ with initial condition $N(0) = 150$
- Substitute $N = 0$ and solve for *t*: $t = 3.96966$ ✓

Part d. i.:

$\frac{dN}{dt} = 2N\left(1 - \frac{N}{600}\right) - 100$ with initial condition $N(0) = 150$

Phase diagram (plot of $\frac{dN}{dt}$ versus *N*)**:**



$\frac{dN}{dt} > 0$ at $t = 0$ ($N = 150$) therefore N initially increases.

$\frac{dN}{dt} > 0$ as N increases ($t > 0$) therefore N continues to increase.

$N \to 300 + 100\sqrt{6}$ as $\frac{dN}{dt} \to 0$.

Valid explanation that refers to a phase diagram. $N \to 300 + 100\sqrt{6}$	**1 mark**

Answer: 545	**1 mark**

Check (use a CAS):

- Solve $\frac{dN}{dt} = 2N\left(1 - \frac{N}{600}\right) - 100$ with initial condition $N(0) = 150$
- Calculate $\lim_{t \to +\infty} N(t)$: $300 + 100\sqrt{6}$

Part d. ii.:

$\frac{dN}{dt}=2N\left(1-\frac{N}{600}\right)-100$ has a turning point at $N=300$ therefore there is a point of inflection at $N=300$

Answer: $N=300$	**1 mark**

Check (use a CAS):

- Solve $\frac{dN}{dt}=2N\left(1-\frac{N}{600}\right)-100$ with initial condition $N(0)=150$
- Solve $\frac{d^2N}{dt^2}=0$ for *t*: $t=\frac{1}{2}\frac{\sqrt{3}}{2}\log_e\left(\frac{4+\sqrt{6}}{4-\sqrt{6}}\right)$
- Substitute $t=\frac{1}{2}\frac{\sqrt{3}}{2}\log_e\left(\frac{4+\sqrt{6}}{4-\sqrt{6}}\right)$ into *N*: $N=300$

Part e.:

- Use a CAS to solve $\frac{dN}{dt}=2N\left(1-\frac{N}{600}\right)-100$ with initial condition $N(0)=150$
- Substitute $N=400$ and solve for t (round to two decimal places):

Answer: $t=1.40$	**1 mark**

Question 5

Part a.:

If $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are normal to each other then $\underset{\sim}{v} \cdot \underset{\sim}{w} = 0$:

$\left(-\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) \cdot \left(3\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}\right) = -3 + 4 - 1 = 0$	**1 mark** Explicit calculation is required.

Part b.:

Method 1:

- A cartesian equation of the plane is $ax + by + cz = d$ where $\underset{\sim}{n} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ is a vector normal to the plane.

  A normal vector to the plane is $\underset{\sim}{n} = \underset{\sim}{v} \times \underset{\sim}{w}$. From a CAS or 'by hand':

  $\underset{\sim}{v} \times \underset{\sim}{w} = 4\underset{\sim}{i} + 2\underset{\sim}{j} + 8\underset{\sim}{k}$

Normal vector: $4\underset{\sim}{i} + 2\underset{\sim}{j} + 8\underset{\sim}{k}$	**1 mark**

- Therefore $4x + 2y + 8z = d$

Substitute the point $(2,\ -2,\ -1)$ into $4x + 2y + 8z = d$ and solve for d: $d = -4$

Therefore $4x + 2y + 8z = -4$

• Explicit solution for *d* using the point $(2,\ -2,\ -1)$ • **Answer:** $2x + y + 4z = -2$	**1 mark** Accept all equivalent answers.

Method 2:

- $(2,\ -2,\ -1)$ is a point contained in the plane Π and the vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are parallel to Π

Therefore a vector equation of Π is $\underset{\sim}{r}_{\Pi} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k} + \lambda\underset{\sim}{v} + \mu\underset{\sim}{w}$ where $\lambda \in R$ and $\mu \in R$

Therefore a set of parametric equations defining Π is

$x = 2 - \lambda + 3\mu$ …. (1) $y = -2 - 2\lambda - 2\mu$ …. (2) $z = -1 + \lambda - \mu$ …. (3)	**1 mark**

- Use a CAS to solve equations (1), (2) and (3) simultaneously for λ, μ and one of either x, y or z and simplify the 'cartesian' solution.

(This is equivalent to solving two of the equations simultaneously for λ and μ in terms of x, y and z and then substituting those solutions into the third equation).

Answer: $2x + y + 4z = -2$	**1 mark**

Part c.:

From the answer to **Part b.**, it is noted that the point $(4,\ 2,\ -3)$ is contained in the plane Π.

The vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are parallel to Π.

Therefore a vector equation of Π is

$\underset{\sim}{r}_{\Pi} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k} + \lambda\underset{\sim}{v} + \mu\underset{\sim}{w}$ where $\lambda \in R$ and $\mu \in R$	**1 mark** • Symbols different to $\underset{\sim}{r}$ and t must be used in the vector equation. • Parameters must be explicitly defined.

Note:

- A symbol different to $\underset{\sim}{r}$ must be used in the vector equation of Π because $\underset{\sim}{r}$ is the position vector of the object.
- A symbol different to t (such as λ) must be used to represent a parameter in the vector equation of Π because t represents time in the position vector of the object.

Choosing $\lambda = \cos(2t)$ and $\mu = \sin(3t)$ in $\underset{\sim}{r}_{\Pi} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k} + \lambda\underset{\sim}{v} + \mu\underset{\sim}{w}$ defines the position vector of the object. Therefore the points on the path followed by the object are a subset of the points contained in the plane.	**1 mark**

Part d.:

- $\underset{\sim}{r}_{\Pi} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k} + \cos(2t)|\underset{\sim}{v}|\hat{\underset{\sim}{v}} + \sin(3t)|\underset{\sim}{w}|\hat{\underset{\sim}{w}}$ (from **part c.**).

- Since $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are perpendicular to each other (from **part a.**), the unit vectors $\hat{\underset{\sim}{v}}$ and $\hat{\underset{\sim}{w}}$ parallel to the plane Π are analogous to the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ parallel to the xy-plane.

- Therefore the parametric equations defining the path followed by the object in the plane Π are equivalent to the parametric equations

$x = |\underset{\sim}{v}|\cos(2t) = \sqrt{6}\cos(2t)$, $\quad y = |\underset{\sim}{w}|\sin(3t) = \sqrt{14}\sin(3t)$

in the xy-plane.

$x = \sqrt{6}\cos(2t)$, $\quad y = \sqrt{14}\sin(3t)$	**1 mark**

- Substitute $x = \sqrt{6}\cos(2t)$, $y = \sqrt{14}\cos(2t)$, $t_1 = 0$ and $t_2 = 1$

into the arc length parametric formula $\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ and evaluate using a CAS:

Answer: 8.24	**1 mark**

Part e.:

$\underset{\sim}{s} = \underset{\sim}{i} + \underset{\sim}{j} + 6\underset{\sim}{k} + t\left(-\underset{\sim}{i} + 6\underset{\sim}{j} - \underset{\sim}{k}\right)$ is the vector equation of a line.

By inspection a set of parametric equations defining the line is

$x = 1 - t \quad \Rightarrow t = -x + 1$ …. (1)

$y = 1 + 6t \quad \Rightarrow t = \dfrac{y-1}{6}$ …. (2)

$z = 6 - t \quad \Rightarrow t = -z + 6$ …. (3)

Equate equations (1), (2) and (3):

Answer: $-x + 1 = \dfrac{y-1}{6} = -z + 6$ Accept all equivalent forms.	**1 mark**

Part f.:

- It is prudent to check whether or not the line intersects the plane Π (if the line intersects the plane then the distance is equal to zero).

Method 1:

Substitute the parametric equations

$x = 1 - t$, $\quad y = 1 + 6t$, $\quad z = 6 - t$

of the line into the equation $2x + y + 4z = -2$ of Π (from **part b.**) and solve for t:

$2(1-t) + (1+6t) + 4(6-t) = -2 \quad \Rightarrow 27 = -2$

which is inconsistent. Therefore the line does not intersect Π.

Method 2: Determine whether or not the line is parallel to Π.

A vector in the direction of the line is $-\underset{\sim}{i}+6\underset{\sim}{j}-\underset{\sim}{k}$ (by inspection) and a normal vector to the plane is $4\underset{\sim}{i}+2\underset{\sim}{j}+8\underset{\sim}{k}$ (from **part b.**).

$\left(-\underset{\sim}{i}+6\underset{\sim}{j}-\underset{\sim}{k}\right)\cdot\left(4\underset{\sim}{i}+2\underset{\sim}{j}+8\underset{\sim}{k}\right)=0$ therefore the line is parallel to the plane.

Note: The line does not lie in Π since any chosen point on the line does not satisfy the equation of the plane.

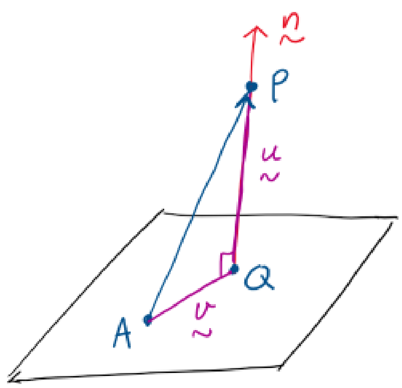
- Calculate the distance.

Note: Since the question is worth 2 marks, "appropriate working **must** be shown".

Method	Example calculation
• Calculate the distance from Π of any chosen point P on the line.	Choose $P(1,\ 1,\ 6)$ (corresponding to $t=0$).
• Define the position vector of P.	$\overrightarrow{OP}=\underset{\sim}{i}+\underset{\sim}{j}+6\underset{\sim}{k}$.
• Get a normal vector $\underset{\sim}{n}$ to Π.	$\underset{\sim}{n}=4\underset{\sim}{i}+2\underset{\sim}{j}+8\underset{\sim}{k}$ (from **part b.**)
• Apply the distance formula $\dfrac{\left\lvert\underset{\sim}{n}\cdot\overrightarrow{OP}-d\right\rvert}{\lvert\underset{\sim}{n}\rvert}$ where d is found from the cartesian equation $ax+by+cz=d$ of Π.	From either a CAS or 'by hand': $\dfrac{\left\lvert\left(4\underset{\sim}{i}+2\underset{\sim}{j}+8\underset{\sim}{k}\right)\cdot\left(\underset{\sim}{i}+\underset{\sim}{j}+6\underset{\sim}{k}\right)-(-2)\right\rvert}{\lvert 4\underset{\sim}{i}+2\underset{\sim}{j}+8\underset{\sim}{k}\rvert}=\dfrac{4\sqrt{7}}{\sqrt{3}}$. **Note:** $d=-2$ comes from $2x+y+4z=-2$ (from **part b.**)

Application of $\dfrac{\left\|\underset{\sim}{\text{n}}\cdot\overrightarrow{OP}-d\right\|}{\|\underset{\sim}{\text{n}}\|}$ **or any other valid method.**	**1 mark**
Answer: $\dfrac{4\sqrt{21}}{3}$	**1 mark**

Derivation of distance formula:



$$|\overrightarrow{PQ}| = |\text{scalar resolute of } \overrightarrow{AP} \text{ in the direction of } \underset{\sim}{n}|$$

$$= \frac{|\overrightarrow{AP}\cdot\underset{\sim}{n}|}{|\underset{\sim}{n}|} = \frac{|(-\overrightarrow{OA}+\overrightarrow{OP})\cdot\underset{\sim}{n}|}{|\underset{\sim}{n}|}$$

$$= \frac{|\underset{\sim}{n}\cdot\overrightarrow{OP} - \underset{\sim}{n}\cdot\overrightarrow{OA}|}{|\underset{\sim}{n}|}$$

$$= \frac{|\underset{\sim}{n}\cdot\overrightarrow{OP} - d|}{|\underset{\sim}{n}|}$$

$\underset{\sim}{n}\cdot\overrightarrow{OA} = d$:

From the vector eq^n of a plane: $\underset{\sim}{r}\cdot\underset{\sim}{n} = \overrightarrow{OA}\cdot\underset{\sim}{n}$

$\Rightarrow ax+by+cz = \overrightarrow{OA}\cdot\underset{\sim}{n}$

$\Rightarrow d = \overrightarrow{OA}\cdot\underset{\sim}{n}$

Question 6

Part a.:

- Let X be the random variable "*Amount of manure (kg) in a small bag*":

$X \sim$ Normal$\left(\mu_X,\ \ \sigma_X = 0.75\right)$

- Let the random variable $W = X_1 + X_2 + X_3$

where X_1, X_2 and X_3 are independent copies of X:

$\mu_W = \mu_{X_1} + \mu_{X_2} + \mu_{X_3} = 3\mu_X$

$\sigma_W^2 = 1^2\sigma_{X_1}^2 + 1^2\sigma_{X_2}^2 + 1^2\sigma_{X_3}^2 = 3\sigma_X^2 = 3(0.75)^2$ therefore $\sigma_W = 0.75\sqrt{3}$

W follows a normal distribution since X_1, X_2 and X_3 are independent normal random variables.

Definition of appropriate random variables. $W \sim$ Normal$\left(3\mu_X,\ \ \sigma_W = 0.75\sqrt{3}\right)$	**1 mark**

- $\Pr(W > 46) = 0.6239$.

$Z = \dfrac{W - \mu_W}{\sigma_W} \qquad \Rightarrow z = \dfrac{46 - 3\mu_X}{0.75\sqrt{3}}$ where $\Pr(Z > z) = 0.6239 \Rightarrow z = -0.31574$

$\dfrac{46 - 3\mu_X}{0.75\sqrt{3}} = -0.31574$	**1 mark** Give 1 mark if z-value is wrong but do not give the answer mark.

- Use a CAS to solve for μ_X: $\mu_X \approx 15.4701$

Answer: $\mu_X = 15.47$	**1 mark** There must be evidence that Z was used.

Rounding check: $\mu_X = 15.47 \Rightarrow \Pr(W > 46) = 0.62385 \approx 0.6239$ ✓

Part b.:

The statistical test is applied at the 3% level of significance therefore the probability of a type I error is $\frac{3}{100} = 0.03$

Answers:	
• $H_0: \quad \mu = 25$ • $H_1: \quad \mu \neq 25$ • 0.03	1 mark

Part c.:

- $\bar{x} = \frac{807.4}{32} = 25.23125$
- Two-sided test therefore $p = 2\Pr\left(\overline{X} > 25.23125 \middle| H_0 \text{ true}\right)$.

Under H_0: $\overline{X} \sim$ Normal$\left(\mu_{\overline{X}} = \mu_X = 25,\ \sigma_{\overline{X}} = \frac{\sigma_X}{\sqrt{n}} = \frac{1}{\sqrt{32}}\right)$

Therefore:

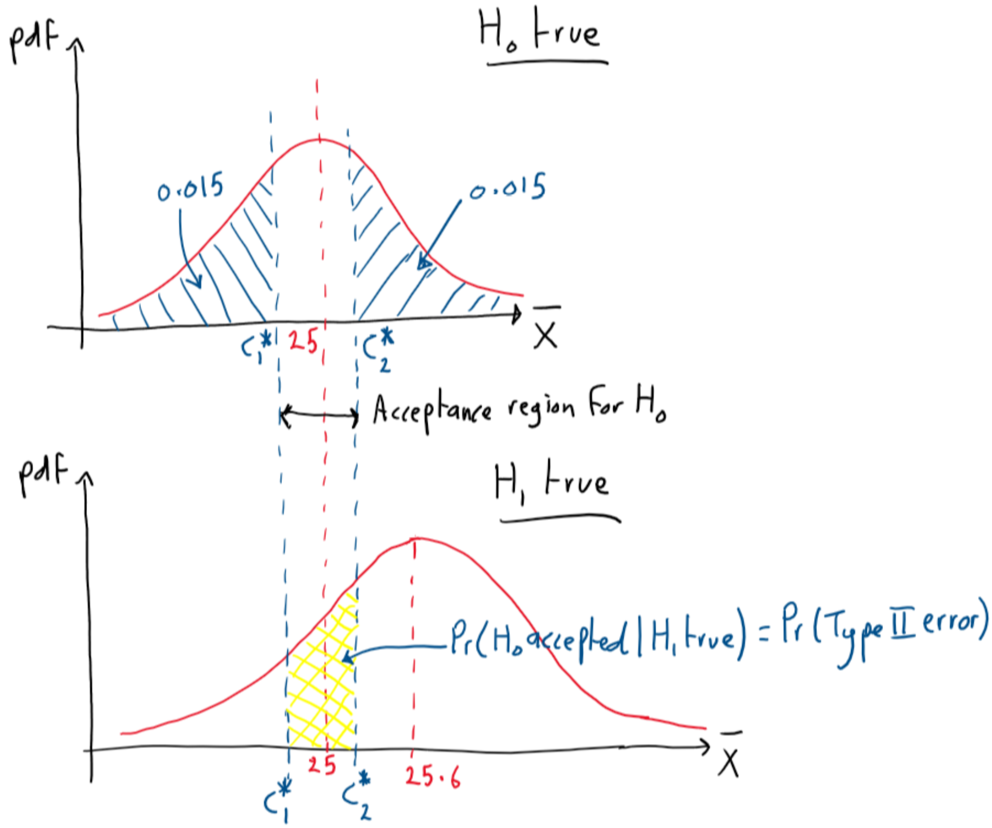
$p = 2\Pr\left(\overline{X} > 25.23125 \middle\| \overline{X} \sim \text{Normal}\left(\mu_{\overline{X}} = 25,\ \sigma_{\overline{X}} = \frac{1}{\sqrt{32}}\right)\right)$ Accept all equivalent statements such as $p = \Pr\left(\left\|\overline{X} - 25\right\| > 0.23125 \middle\| \overline{X} \sim \text{Normal}\left(\mu_{\overline{X}} = 25,\ \sigma_{\overline{X}} = \frac{1}{\sqrt{32}}\right)\right)$	1 mark

- From a CAS:

Answer: 0.0954	1 mark

Part d.:

- Definition: A type II error is the failure to reject H_0 when H_0 is false.
- Critical values C_1^* and C_2^* of $\overline{X}$ (use a CAS):



Note 1:
C_1^* and C_2^* are symmetric about $\mu_{\overline{X}} = 25$ by symmetry of the normal distribution.

Note 2:
The interval $\left(C_1^*,\ C_2^*\right)$ is **not** a confidence interval. It is the acceptance region for H_0

- H_0 is accepted when $\bar{x} \in \left(C_1^*,\ C_2^*\right)$

Option 1:

$\Pr\left(\overline{X} > C_2^*\right) = \frac{1}{2}(0.03) = 0.015 \Rightarrow C_2^* = 25.3836214$. $\Pr\left(\overline{X} < C_1^*\right) = 0.015 \Rightarrow C_1^* = 24.61637859$.

Option 2:

$\Pr\left(25 - k < \overline{X} < 25 + k\right) = 0.97 \Rightarrow k = 0.383621$ $C_1^* = 25 - k$ and $C_2^* = 25 + k$

A correct critical value: Either $C_1^* = 24.61637859$ or $C_2^* = 25.3836214$	**1 mark** More than three decimal place accuracy is required (to avoid rounding error in final answer)
Statement of probability of type II error using critical values: $\Pr\left(C_1^* < \overline{X} < C_2^* \middle\| \mu_{\overline{X}} = 25.6\right)$	**1 mark**
Answer: 0.110	**1 mark**