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Student Name:………………………………………

SPECIALIST MATHEMATICS UNITS 3 & 4

TRIAL EXAMINATION 1

2024

Reading Time: 15 minutes
Writing time: 1 hour

Instructions to students

This exam consists of 10 questions.
All questions should be answered in the spaces provided.
There is a total of 40 marks available.
The marks allocated to each of the questions are indicated throughout.
Students may **not** bring any notes or calculators into the exam.
Where more than one mark is allocated to a question, appropriate working must be shown.
An exact answer is required to a question unless otherwise specified.
Unless otherwise indicated, diagrams in this exam are not drawn to scale.
A formula sheet can be found at the end of this exam.

Question 1 (3 marks)

Evaluate $\int_{\frac{1}{2}}^{1} \frac{\log_e(2x)}{x^2}\,dx$.

Question 2 (3 marks)

Solve the equation $8\sin^4(x)\cot^4(x) + \cos(2x) + 12\sin^2(x) = 8$ for $x \in \left(\frac{\pi}{2}, \pi\right)$.

Question 3 (4 marks)

A soft serve ice cream machine automatically dispenses ice cream into a bowl, then dispenses chocolate topping into the same bowl. The volume of ice cream dispensed is normally distributed with a mean of 125 mL and a standard deviation of 4 mL. The volume of chocolate topping dispensed is normally distributed and has a mean of 20 mL and a standard deviation of 3 mL. The volume of chocolate topping dispensed is independent of the volume of ice cream dispensed.

a. What is the probability that a bowl of ice cream with chocolate topping has a total volume more than 155 mL? Use $\Pr(-2 < Z < 2) = 0.9545$, where Z is the standard normal variable and give your answer correct to three decimal places. 2 marks

b. If the cost of ice cream is \$11 per litre and the cost of chocolate topping is \$15 per litre, what is the mean total cost, correct to the nearest cent, of a bowl of ice cream with chocolate topping? 2 marks

Question 4 (3 marks)

Prove using mathematical induction that $7+9+11+....+(2n+1)=n^2+2n-8$ for all integers $n \geq 3$.

Question 5 (3 marks)

Let $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j} + \underset{\sim}{k}$, where $m \in Z$.

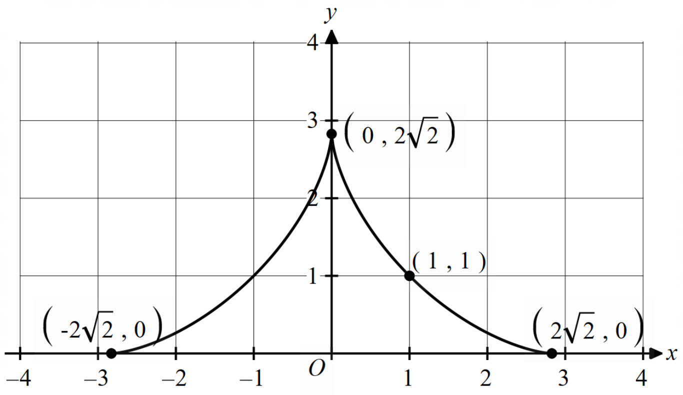
The scalar resolute of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}$ is $\dfrac{7}{\sqrt{14}}$.

a. Find the value of m. 1 mark

b. Find the vector resolute of $\underset{\sim}{b}$ that is perpendicular to $\underset{\sim}{a}$. 2 marks

Question 6 (6 marks)

The part of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ for $y \geq 0$ is shown below.



a. Find the gradient of the curve at the point (1, 1). 2 marks

The curve can also be described parametrically with the equations

$$x = 2^{\frac{3}{2}}\cos^3(t)$$
$$y = 2^{\frac{3}{2}}\sin^3(t) \qquad \text{for } 0 \le t \le \pi$$

b. Find the surface area obtained by rotating the curve shown about the x-axis. 4 marks

Question 7 (3 marks)

Solve $2\overline{z}^2 + 2z + \text{Re}(z) = -5$ for $z \in C$.

Question 8 (4 marks)

Solve the differential equation $\frac{dv}{dt} = \frac{2v}{1+t^2}$, given that $v(0) = e$.
Give your answer in the form $v = f(t)$.

Question 9 (4 marks)

The vector equations of two lines are as follows:

$$\underset{\sim}{r}_1(\lambda) = 10\underset{\sim}{i} - \sqrt{2}\underset{\sim}{j} + \frac{5\sqrt{2}}{4}\underset{\sim}{k} + \lambda(\sqrt{2}\underset{\sim}{i} + a\underset{\sim}{j} + 3\underset{\sim}{k})$$

$$\underset{\sim}{r}_2(\mu) = 5\underset{\sim}{i} + \sqrt{2}\underset{\sim}{j} - 5\sqrt{2}\underset{\sim}{k} + \mu(\sqrt{2}\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$$

where $\mu, \lambda, a \in R$.

a. If the angle between the two lines is $60°$, show that $a = -\frac{7}{5}$. 2 marks

b. Hence, find the Cartesian equation of the plane that contains the lines $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$. 2 marks

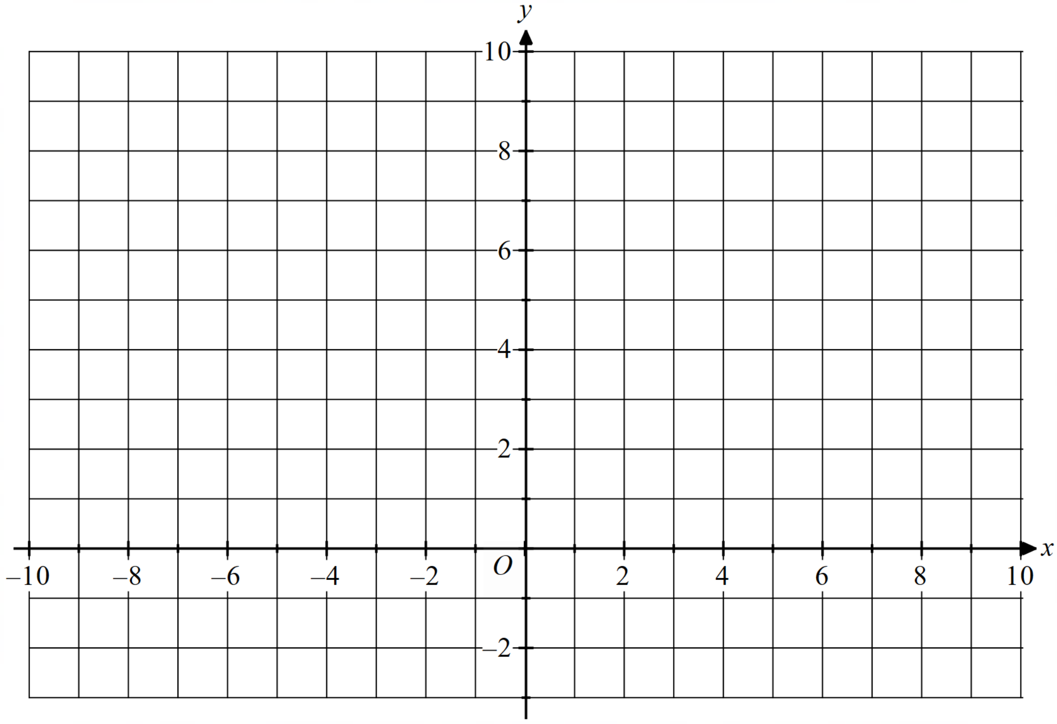
Question 10 (7 marks)

Consider the function $f(x)=\dfrac{2x^2-4}{(x-2)^2}$.

a. Show that $f(x)$ has a stationary point of at $(1,-2)$. 2 marks

b. Show that $f(x)$ has a point of inflection at $x=\dfrac{1}{2}$. 2 marks

c. Sketch the graph of $y = f(x)$ on the set of axes below. Label any asymptotes with their equations and any axis intercepts and the stationary point with their coordinates. There is no need to label the coordinates of the point of inflection. 3 marks



Specialist Mathematics formula sheet

Mensuration

area of a circle segment	$\frac{r^2}{2}(\theta - \sin(\theta))$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$	sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
volume of a pyramid	$\frac{1}{3}Ah$	cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = r\text{cis}(\theta)$	$\lvert z\rvert = \sqrt{x^2 + y^2} = r$	
$-\pi < \text{Arg}(z) \le \pi$	$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$	
$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$	de Moivre's theorem	$z^n = r^n \text{cis}(n\theta)$

Data analysis, probability and statistics

for independent random variables $X_1, X_2 \ldots X_n$	$\mathrm{E}(aX_1 + b) = a\mathrm{E}(X_1) + b$ $\mathrm{E}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1\mathrm{E}(X_1) + a_2\mathrm{E}(X_2) + \ldots + a_n\mathrm{E}(X_n)$	
	$\mathrm{Var}(aX_1 + b) = a^2\mathrm{Var}(X_1)$ $\mathrm{Var}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1^2\mathrm{Var}(X_1) + a_2^2\mathrm{Var}(X_2) + \ldots + a_n^2\mathrm{Var}(X_n)$	
for independent identically distributed variables $X_1, X_2 \ldots X_n$	$\mathrm{E}(X_1 + X_2 + \ldots + X_n) = n\mu$	
	$\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = n\sigma^2$	
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$	
distribution of sample mean $\bar{X}$	mean	$\mathrm{E}(\bar{X}) = \mu$
	variance	$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \log_e\lvert x\rvert + c$
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$	$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$	$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$	$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$
$\frac{d}{dx}\left(\cot(ax)\right) = -a\operatorname{cosec}^2(ax)$	$\int \operatorname{cosec}^2(ax)\,dx = -\frac{1}{a}\cot(ax) + c$
$\frac{d}{dx}\left(\sec(ax)\right) = a\sec(ax)\tan(ax)$	$\int \sec(ax)\tan(ax)\,dx = \frac{1}{a}\sec(ax) + c$
$\frac{d}{dx}(\operatorname{cosec}(ax)) = -a\operatorname{cosec}(ax)\cot(ax)$	$\int \operatorname{cosec}(ax)\cot(ax)\,dx = -\frac{1}{a}\operatorname{cosec}(ax) + c$
$\frac{d}{dx}\left(\sin^{-1}(ax)\right) = \frac{a}{\sqrt{1-(ax)^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\frac{d}{dx}\left(\cos^{-1}(ax)\right) = \frac{-a}{\sqrt{1-(ax)^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\frac{d}{dx}\left(\tan^{-1}(ax)\right) = \frac{a}{1+(ax)^2}$	$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
	$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ n \neq -1$
	$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e\lvert ax+b\rvert + c$

Calculus – continued

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
integration by parts	$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$
Euler's method	If $\frac{dy}{dx} = f(x, y), x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h \times f(x_n, y_n)$
arc length parametric	$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area Cartesian about x-axis	$\int_{x_1}^{x_2} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$
surface area Cartesian about y-axis	$\int_{y_1}^{y_2} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$
surface area parametric about x-axis	$\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area parametric about y-axis	$\int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$	
constant acceleration formulas	$v = u + at$	$s = ut + \frac{1}{2}at^2$
	$v^2 = u^2 + 2as$	$s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$	$\lvert\underset{\sim}{r}(t)\rvert = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$
	$\dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
for $\underset{\sim}{r}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{r}_2 = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$	vector scalar product $\underset{\sim}{r}_1 . \underset{\sim}{r}_2 = \lvert\underset{\sim}{r}_1\rvert\lvert\underset{\sim}{r}_2\rvert\cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$
	vector cross product $\underset{\sim}{r}_1 \times \underset{\sim}{r}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1)\underset{\sim}{i} + (x_2z_1 - x_1z_2)\underset{\sim}{j} + (x_1y_2 - x_2y_1)\underset{\sim}{k}$
vector equation of a line	$\underset{\sim}{r}(t) = \underset{\sim}{r}_1 + t\underset{\sim}{r}_2 = (x_1 + x_2t)\underset{\sim}{i} + (y_1 + y_2t)\underset{\sim}{j} + (z_1 + z_2t)\underset{\sim}{k}$
parametric equation of a line	$x(t) = x_1 + x_2t \quad y(t) = y_1 + y_2t \quad z(t) = z_1 + z_2t$
vector equation of a plane	$\underset{\sim}{r}(s,t) = \underset{\sim}{r}_0 + s\underset{\sim}{r}_1 + t\underset{\sim}{r}_2$ $= (x_0 + x_1s + x_2t)\underset{\sim}{i} + (y_0 + y_1s + y_2t)\underset{\sim}{j} + (z_0 + z_1s + z_2t)\underset{\sim}{k}$
parametric equation of a plane	$x(s,t) = x_0 + x_1s + x_2t, \quad y(s,t) = y_0 + y_1s + y_2t, \quad z(s,t) = z_0 + z_1s + z_2t$
Cartesian equation of a plane	$ax + by + cz = d$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$	$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$
$\sin(2x) = 2\sin(x)\cos(x)$	
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$
$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$	$\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$

End of formula sheet